

The Real and Complex Approaches to a Class of Dirichlet's Sinc Integrals

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Abstract. Integration is a powerful tool to solve many problems in different areas, such as probability theory, physics, economics, etc. This paper deals with the integration of an important function, Dirichlet's sinc functions of the form $\sin x / x$. Sinc function is a significant function that is used in many different fields, particularly in engineering and optics. In this article, by using the iteration method, a general solution about the integrals of the sinc functions with different powers is found, and several examples are given to support this idea. Meanwhile, Cauchy's residue theorem and contour integration are the fundamental methods in the complex analysis, and they are chosen to compute the integrals of sinc functions with power one, two, and three. The integrals found in the examples correspond to the general solutions. The findings of the paper suggest that the real methods have the advantage to discover the recursion formula, while the complex methods are powerful in dealing with specific integrals.

Keywords: Cauchy's residue theorem; Contour integration; Sinc function; Recursion formula.

1. Introduction

Over the years, many special functions are extensively studied in various disciplines. In particular, Dirichlet's sinc function has been significantly used in many aspects. The first remarkable application is the sinc approximation. A variety of numerical methods based on the sinc approximation have been developed during recent decades [1]. It deals with problems that have solutions including singularities, infinite domains, or boundary layers, and is omnipresent for estimating every operation in calculus [2]. Another application is the Fourier transformation of the sinc function. This is commonly used in digital signal processing as an idealized filter that filters out the frequency components that are above a given cutoff frequency [3,4]. Overall, there are two different forms of sinc function, normalized and unnormalized. The normalized function is widely used in engineering, while the unnormalized form is more used in mathematics [5].

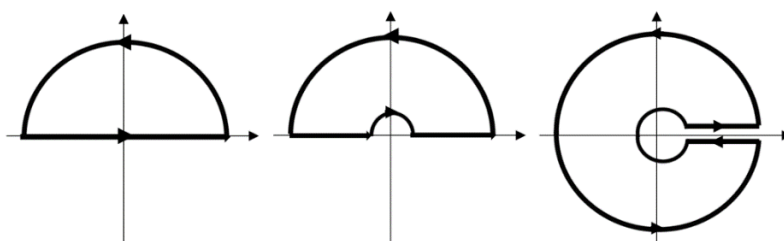


Fig 1. Three common contours

Cauchy's residue theorem is the main theorem used during the integrations in this paper [6]. It is applied when evaluating an integral of an analytic function over a simple closed contour, and the final answer can be found by multiplying $2\pi i$ by the sum of the residues of $f(z)$ at the singularities inside the contour, i.e.,

$$\oint f(z)dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=s_k} f(z). \quad (1)$$

An appropriate contour needs to be constructed in order to find the integrals. Figure 1 illustrates three common contours used when doing contour integration [7]. It should be chosen based on the bounds of the integral, singularities of the function, the parity of the integrand, etc. Since the sinc function has a singularity at $z = 0$, In the following examples the second contour is chosen.

2. General solutions of $\int_{-\infty}^{\infty} \frac{\sin^m x}{x^n} dx$

The aim of this section is to find the general solutions of [8]

$$I(m, n) = \int_{-\infty}^{\infty} \frac{\sin^m x}{x^n} dx, \quad (2)$$

where m and n are both integers. Depending on the parity, the calculation procedure is different and is discussed below.

2.1. Even parity

Consider m and n are both even numbers, then

$$\sin^{2m} x = \frac{1}{2^{2m-1}(-1)^m} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} \cos(2m - 2k)x + \frac{1}{2^{2m}} \binom{2m}{m}. \quad (3)$$

Now, its high derivatives can be easily found. By differentiating the expression $(2n - 1)$ -times, it follows that

$$(\sin^{2m} x)^{(2n-1)} = \frac{(-1)^{m-n}}{2^{2m-1}} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} (2m - 2k)^{2n-1} \sin(2m - 2k)x. \quad (4)$$

Using integration by parts $(2n - 1)$ times, $I(m, n)$ becomes

$$\int_{-\infty}^{\infty} \frac{\sin^m x}{x^n} dx = \frac{(-1)^{m-n}\pi}{2^{2m}(2n-1)!} \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} (2m - 2k)^{2n-1}. \quad (5)$$

2.2. Odd parity

For m and n to be both odd numbers, $\sin^{2m-1} x$ can be written as

$$\sin^{2m-1} x = \frac{1}{2^{2m}(-1)^m} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} \sin(2m + 1 - 2k)x. \quad (6)$$

By using a similar method, differentiating $\sin^{2m-1} x$ $(2n)$ -times,

$$(\sin^{2m-1} x)^{(2n)} = \frac{(-1)^{m-n}}{2^{2m}} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} (2m + 1 - 2k)^{2n} \sin(2m + 1 - 2k)x \quad (7)$$

Therefore, $\int_0^{\infty} \frac{\sin^{2m-1} x}{x^{2n+1}} dx$ is equal to integrating $\frac{(\sin^{2m+1} x)^{2n}}{(2n)!x}$ $2n$ times. Thus,

$$\int_0^{\infty} \frac{\sin^{2m-1} x}{x^{2n+1}} dx = \frac{(-1)^{m-n}\pi}{2^{2m+1}(2n)!} \sum_{k=0}^m (-1)^k \binom{2m+1}{k} (2m + 1 - 2k)^{2n}. \quad (8)$$

Specially, when m and n are the same, a general solution is obtained as follow:

$$\int_{-\infty}^{\infty} \frac{\sin^n x}{x^n} dx = \frac{2\pi}{2^n(n-1)!} \sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{k} (n - 2k)^{n-1}. \quad (9)$$

When $n = 1, 2, 3$, it is easy to check that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi, \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx = \pi, \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx = \frac{3\pi}{4}. \quad (10)$$

3. Examples

3.1. The Case of $n = 1$: $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

In order to compute the integral $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx$, a closed contour that is constructed by two semicircles with radius R and radius ε respectively, as shown in Fig. 2, is chosen.

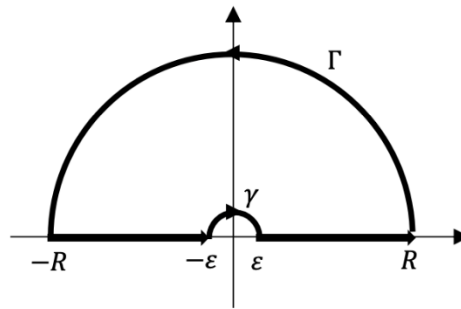


Fig 2. The keyhole-like contour of $f(z)$

Calculating the integral when $R \rightarrow \infty$ and $\varepsilon \rightarrow 0$. Then, the integral can be divided into four parts,

$$\oint_{\Gamma} f(z) dz = \int_{-R}^{-\varepsilon} f(x) dx + \int_{\gamma} f(z) dz + \int_{\varepsilon}^R f(x) dx + \int_{\Gamma} f(z) dz, \quad (11)$$

where $f(z) = e^{iz}/z$. For the integral $\int_{-R}^{-\varepsilon} f(x) dx + \int_{\varepsilon}^R f(x) dx$, it is found that [9]

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x} dx + \int_{\varepsilon}^R \frac{e^{ix}}{x} dx = \int_{\varepsilon}^R \frac{e^{ix}}{x} dx - \int_{\varepsilon}^R \frac{e^{-ix}}{x} dx = \int_{\varepsilon}^R \frac{e^{ix} - e^{-ix}}{x} dx = 2i \int_0^{\infty} \frac{\sin x}{x} dx. \quad (12)$$

Next, for the integral of the smaller semicircle $\int_{\gamma} \frac{e^{iz}}{z} dz$, it is useful to introduce $\gamma(t) = \varepsilon e^{i(\pi-t)}$, where $0 \leq t \leq \pi$. Its derivative satisfies $\gamma'(t) = -i\varepsilon e^{i(\pi-t)} = -i\gamma(t)$. After doing the substitution, the integral becomes

$$\int_{\gamma} f(z) dz = \int_0^{\pi} \frac{e^{i\gamma(t)}}{\gamma(t)} (-i\gamma(t)) dt = i \int_0^{\pi} -e^{i\varepsilon e^{i(\pi-t)}} dt \rightarrow -\pi i. \quad (13)$$

as $\varepsilon \rightarrow 0$. Finally, to find the integral along the contour Γ , let $\Gamma(t) = Re^{it}$ where $0 \leq t \leq \pi$. Then $\Gamma'(t) = iRe^{it} = i\gamma(t)$. Hence,

$$\int_{\Gamma} f(z) dz = \int_0^{\pi} \frac{e^{i\gamma(t)}}{\gamma(t)} i\gamma(t) dt = i \int_0^{\pi} e^{iRe^{it}} dt. \quad (14)$$

Using the integral inequality, $\left| \int_0^{\pi} e^{iRe^{it}} dt \right| = \left| \int_0^{\pi} e^{iR \cos t - R \sin t} dt \right| \leq \int_0^{\pi} |e^{iR \cos t}| |e^{-R \sin t}| dt = \int_0^{\pi} |e^{-R \sin t}| dt$. When $R \rightarrow \infty$, $\int_0^{\pi} |e^{-R \sin t}| dt \rightarrow 0$, for $0 \leq t \leq \pi$. Since $\left| \int_0^{\pi} e^{iRe^{it}} dt \right| \leq 0$, therefore $\left| \int_0^{\pi} e^{iRe^{it}} dt \right| = 0$. Then, by applying the Cauchy's integral theorem, $\oint_{\Gamma} f(z) dx = 2i \int_{\varepsilon}^R \frac{\sin x}{x} dx - \pi i = 0$. Taken together, it turns out that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi, \quad (15)$$

which is the same as that of Eq. (10).

3.2. The Case of $n = 2$: $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$

To evaluate the integral $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx$, the same contour shown in Fig. 2 is used, and the integral is divided into four as Eq. (11). Moreover, $\frac{\sin^2 x}{x^2}$ can be changed into the exponential form $\frac{2 - e^{2ix} - e^{-2ix}}{4x^2}$. First, by combining the first two parts using substitution,

$$\int_{-R}^{-\varepsilon} f(x) dx + \int_{\varepsilon}^R f(x) dx = \int_{\varepsilon}^R \frac{2 - e^{2ix} - e^{-2ix}}{4x^2} dx = 2 \int_{\varepsilon}^R \frac{\sin^2 x}{x^2} dx = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx. \quad (16)$$

For the part $\int_{\gamma} f(z) dz$, Cauchy's residue theorem is used here. Since $\frac{\sin^2 x}{x^2}$ is equal to the real part of $\frac{1 - e^{2ix}}{2x^2}$, it can be expanded into $-\frac{i}{x} + i + \frac{2ix}{3} - \frac{x^2}{3} - \frac{2ix^3}{15} + \dots$. Thus, the residue of $\frac{\sin^2 x}{x^2}$ at

$x = 0$ is equal to $-i$. According to the theorem, the integral is equal to $-\pi i \operatorname{Res}_{x=0} f(x) = -\pi$ as a semicircle contour from π to 0 is used. Last, by using the Jordan's Lemma, $\int_{\Gamma} f(z) dz = 0$. Since, $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx - \pi + 0 = 0$, the final integral is equal to π , which is again the same as that of Eq. (10).

3.3. The Case of $n = 3$: $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx$

To compute the integral, $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx$ with a power of three, a similar method and the same contour is used as previously mentioned. Since trigonometry is hard to integral, so by doing some transformation, $\frac{\sin^3 x}{x^3}$ is equal to the imaginary part of $\frac{e^{3ix} - 3e^{ix}}{-4x^3}$. The integral is also done separately by four parts as Eq. (11). For the first two parts, by using the substitution $u = -x$ [10],

$$\int_{-R}^{-\varepsilon} \frac{e^{3ix} - 3e^{ix}}{-4x^3} dx + \int_{\varepsilon}^R \frac{e^{3ix} - 3e^{ix}}{-4x^3} dx = 2i \int_0^{\infty} \frac{\sin^3 x}{x^3} dx. \quad (17)$$

For the third part, integrating along the smaller semicircle with radius ε , Cauchy's residue theorem is employed. Since only the imaginary part of the $f(z)$ is taken, $\int_{\gamma} f(z) dz$ can be written as $\int_{\gamma} \frac{e^{3iz} - 3e^{iz}}{-4z^3} - \frac{1}{2z^3} dz$. Then expression $\frac{e^{3iz} - 3e^{iz}}{-4z^3} - \frac{1}{2z^3}$ can be expanded into $\frac{3}{4z} + i - \frac{13z}{16} + \dots$. This cancels out the pole of order 3, and now the residue is $3/4$, the coefficient of the term with power of negative one, when z equals to zero. As the integral is computed along the semicircle from π to 0 , so the integral is equal to multiplying the residue by $-\pi i$. Thus, $\int_{\gamma} f(z) dz = -\frac{3}{4}\pi i$. Finally, for the last part of the integration, by using the Jordan's Lemma, $\int_{\Gamma} f(z) dz = 0$. Due to the Cauchy's integral theorem, $i \int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx - \frac{3}{4}\pi i = 0$. Thus,

$$\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3} dx = \frac{3}{4}\pi, \quad (18)$$

which is the same as that of Eq. (10).

4. Conclusion

To summarize, the residue theorem is an efficient method to conquer a lot of improper integrals. To make integrations more convenient to calculate, adding or subtracting certain terms can cancel out the singularities without changing their magnitude, so only the first-order pole needs to be considered. Otherwise, it will be much more complicated to compute the high-order pole. With the help of Cauchy's residue theorem, the integral can be easily calculated by finding the residue using the Laurent series. The residue theorem is also frequently combined with Jordan's lemma to access contour integrals and improper integrals, which is applied when evaluating the integral along the larger semi-circle contour. Since the improper integral $\frac{\sin^n x}{x^n}$ converges, other methods of finding the integral may include using the holomorphic function, Feynman's technique, Laplace transform, etc. By contrast, using the contour integration and the residue theorem seems to be the most efficient way. It is important to choose an appropriate contour and find the correct residue at the singularities. Overall, the results of the examples calculated are found to satisfy the general solution.

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