

Evaluation of Limits Involving Trigonometric Functions by L'Hospital Rule

Mingrui Sun *

High school attached to Northeast Normal University, Changchun, 130021, China

* Corresponding Author Email: 1700210732@stu.sqxy.edu.cn

Abstract. Limit is the basic idea of calculus and it is the theoretical basis of a series of important concepts such as derivatives, continuity, definite integral, etc., In addition, it is a tool for solving many problems, thus correctly mastering the calculation methods and skills of the limit is of great significance to learn more advanced mathematics well. Through induction and conclusion, this article mainly introduces the application of several methods of solving the limit in light of L'Hopital's rule, as well as gives the analysis of example problems for each method. For this purpose, it is expected that readers can get some inspiration for solving problems from them, so that they can solve problems more clearly. Furthermore, a brief introduction is made to the main content of the L'Hospital's rule in advanced mathematics and an analysis is given of the misunderstandings in the teaching and application of the L'Hospital's rule, and a detailed description of many specific application of the L'Hospital's rule in the analysis of limit problems.

Keywords: L'Hopital's rule; Limit; Taylor expansion; Trigonometric functions.

1. Introduction

The limit of the infinitive is a common and important type of limit [1]. In 1821, French mathematician Cauchy introduced the concept of limit, proposed the method of defining limit in calculus, and used inequality to describe the process of limit. Cauchy did the most systematic pioneering work on the definite integral. Later, through the improvement of German mathematician Wilstras, a strict definition of calculus, namely language, was gradually developed. So far, the basic concepts of calculus have been strictly discussed, and Cauchy's explanation of calculus by using limits has been widely used by later mathematicians [2].

Limit is an important concept in advanced mathematics. Many concepts in advanced mathematics are defined in limit language. For example, continuity of functions, derivatives, definite integrals, etc. Limit is a very important idea. In real life Many problems that cannot be quantified, such as the area and perimeter of irregular figures, can be solved by seeking limits [3]. In the process of learning advanced mathematics, the calculation of limits has always been the focus and difficulty of our learning, mainly because the subject of the limit is flexible and changeable, and the methods are also varied, such as the definition method, the zero-factor elimination method, the infinite mass subtraction method, and the L'Hopital's rule. Of course, for some complex problems, there is a certain degree of difficulty in solving them in a short time. For different questions, if the method chosen is inappropriate, the questions will not be solved. For the limit calculation of some complex questions, it is found that the calculation process using Taylor's formula is simple and not easy to make mistakes. Therefore, it is desirable to discuss some researches of Taylor's formula in function limit operation [4].

The L'Hopital's method has an important application in solving the limit of infinitives. Also, L'Hopital's rule is a problem-solving method for deriving both numerator and denominator under certain conditions, finding the limit, and finally obtaining the value of the infinitive. According to the current level of cognition, limit of ratio of two infinities or limit of ratio of two infinities may exist, but it may not exist. Therefore, when solving such infinitive limits, it is necessary to convert them into important limit forms or use the expression form of limit arithmetic to carry out calculations, and L'Hopital's method is a general way to solve such infinitive limits properly [5]. The application

of L'Hopital's rule can transform limit operation of infinitives into simpler derivative operations, which provides a relatively simple and effective calculation method for solving the limit.

2. Methods

2.1. L' Hospital Rule

The famous L'Hopital's rule argues that, if real functions $f(x)$ and $g(x)$ satisfy $\lim_{x \rightarrow x_0} f(x) = \lim_{x \rightarrow x_0} g(x) = 0$, both are differentiable in a specific hollow neighbour of point x_0 , and $\lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)} = A$ (A could be real number, also could be $\pm\infty$), then the limit [6]

$$\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow x_0} \frac{f'(x)}{g'(x)}. \quad (1)$$

There are two different forms of the limit. One is $\frac{0}{0}$ form while the other is $\frac{\infty}{\infty}$ form. For the former case, $\lim_{x \rightarrow x_0} \frac{\tan x}{x^2 \sin x} = \lim_{x \rightarrow x_0} \frac{\tan x - x}{x^3} \times \frac{x}{\sin x} = \lim_{x \rightarrow x_0} \frac{\tan x - x}{x^3} = \lim_{x \rightarrow x_0} \frac{\sec^2 x - 1}{3x^2} = \frac{1}{3}$. Actually, the L'Hopital's rule can also be used to other case such as the " $\infty - \infty$ " undetermined form, where $\lim_{x \rightarrow 0} \frac{1}{\ln(1+x)} - \frac{1}{x}$ is an example. To solve it, it is first to transform it into "0" type, that is $\lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x \ln(1+x)}$. At this time, if the L'Hopital's rule is used directly, derivation of the denominator will also be more complicated. Instead, one can use the equivalent infinitesimal to replace it, and then use L'Hopital's rule to solve is $\lim_{x \rightarrow 0} \frac{1}{\ln(1+x)} - \frac{1}{x} = \lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x \ln(1+x)} = \lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x \rightarrow 0} \frac{1 - \frac{1}{1+x}}{2x} = \frac{1}{2}$.

Before applying L'Hopital's rule to solve the problem, the problem must be verified to determine whether it meets the conditions for using L'Hopital's rule [7]. First, whether the limits of the numerator and denominator are equal to infinity or 0. Second, whether the numerator and denominator can be differentiated within a limited area. If the title satisfies the above two conditions, then a derivative on the function can be performed, and whether there is a limit after the derivative can be judged. If it is still not sure whether there is a limit after derivation, the result is still an infinitive and needs to be verified until it meets the conditions before using L'Hopital's rule. Imposing conditions will lead to errors in the calculation results. An Indeterminate form of any algebraic expression with a limit applied to it corresponds to a solution that is not viable. Thus, it is of the form zero over zero, zero over infinity, and so on. Therefore, it does not provide insight into the problem appropriately.

2.2. Taylor's Expansion

When hunting limit of a given function, the equivalent infinitesimal is used instead of using only the first-order property. L'Hopital's rule can only be reduced by one order at a time, and Taylor's formula can be used in one step.

Taylor's formula is an important content of calculus theory. The main idea is to use simple polynomials to approximate and express more complex functions. It has a wide range of applications in solving function limits, inequalities, and approximate calculations. It solves the problem of using differential calculations [8]. The problem of low incremental accuracy also provides people with an error estimation formula and realizes an effective control of the error. This paper focuses on the application of Taylor expansion to deduce the equivalent infinitesimal of the algebraic sum of functions, so as to explore. The equivalent infinitesimal substitution is often not applicable to the algebra of functions and the essence of seeking limits.

3. Applications

The first question is [9]

$$L = \lim_{x \rightarrow \infty} (e^x + x)^{\frac{1}{x}} \quad (2)$$

First, let $x \rightarrow \infty$, then $(e^x + x)^{\frac{1}{x}} \rightarrow \infty^0$. Define $z = (e^x + x)^{\frac{1}{x}}$ and take the logarithmic of both sides. After a little simplification, it is found that

$$\ln z = \ln(e^x + x)^{\frac{1}{x}} = \frac{\ln(e^x + x)}{x}. \quad (3)$$

By applying L'Hospital's rule, it is found that

$$\lim_{x \rightarrow \infty} \ln z = \lim_{x \rightarrow \infty} \frac{\ln(e^x + x)}{x} = \lim_{x \rightarrow \infty} \frac{\frac{e^x + 1}{e^x + x}}{1} = \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x} \quad (4)$$

The limit behaves like $\frac{e^x + 1}{e^x + x} \rightarrow \frac{\infty}{\infty}$ as $x \rightarrow \infty$, and thus the L'Hospital rule can be used again. This gives $\lim_{x \rightarrow \infty} \ln z = \lim_{x \rightarrow \infty} \frac{e^x + 1}{e^x + x} = \lim_{x \rightarrow \infty} \frac{e^x}{e^x} = 1$. Notice that the L'Hospital's rule is used twice here and some simplification are employed so as to calculate the limit easily.

The second question is

$$L = \lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}. \quad (5)$$

First of all, the first step to be done is to verify that the L'Hospital rule actually works with this restriction. Although it seems like a stupid step, readers really need to get in the habit of checking if it works before applying it. If the L'Hospital's rule is applied to a problem to which it cannot be applied, then people will almost certainly get the wrong answer. For the question under consideration, $\frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} \rightarrow \frac{0}{0}$ as $x \rightarrow 2$. Therefore, this is a proper form that permits the use of L'Hospital's rule. So, by applying the rule, it is found that

$$L = \lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6} = \lim_{x \rightarrow 2} \frac{3x^2 - 14x + 10}{2x + 1} = -\frac{6}{5}. \quad (6)$$

The third question is

$$L = \lim_{x \rightarrow 0} x \sqrt{\frac{\tan x}{x}}. \quad (7)$$

By using of the fact that $u = e^{\ln u}$, then Eq. (12) is equivalent to

$$L = e^{\lim_{x \rightarrow 0} \log x \sqrt{\frac{\tan x}{x}}} = e^{\lim_{x \rightarrow 0} \frac{1}{x} \log \frac{\tan x}{x}} = e^{L'}. \quad (8)$$

To further calculate the limit in light of L'Hospital's rule, it is useful to multiply two formulas together, which means that

$$L' = \lim_{x \rightarrow 0} \frac{\log(1 + \frac{\tan x}{x})}{\frac{\tan x}{x} - 1} \times \frac{\frac{\tan x}{x} - 1}{x} = \lim_{x \rightarrow 0} 1 \times \frac{\tan x - x}{x^2}. \quad (9)$$

After turning $\tan(x)$ into $\cos(x)$ and $\sin(x)$, the final result

$$L' = \frac{\sin x - x \cos x}{x^2 \cos x} = \lim_{x \rightarrow 0} \frac{\sin x - x \cos x}{x^2 \times 1} = \lim_{x \rightarrow 0} \frac{\sin x - x}{x^2} + x \times \frac{1 - \cos x}{x^2} = 0. \quad (10)$$

Hence, the limit $L = e^{L'} = 1$.

When seeking the limit of the infinitive, the logarithmic property of $e^{\ln a} = a$ can be used to convert the function in the title into an exponential function with e as the base number, and then continue to find the limit. In addition, different methods can be used to replace the functions in the

topic for different topics, such as $\ln(1+x) \sim x$, when x tends to 1 infinitely from the positive axis, and the equivalent infinitesimal replacement can be used. The formula replaces $\ln(1+x)$ with x and continues to be solve.

The fourth question is [10]

$$L = \lim_{x \rightarrow \infty} \frac{e^x}{\left(1 + \frac{1}{x}\right)^{x^2}}. \quad (11)$$

Expressions for numerators and denominators of rational functions are formed exponentially. Therefore, it is helpful to use the exponential limit rule to calculate limits of rational functions. The expression in denominator is similar to the exponential limit rule. Therefore, it is desirable to prepare the given function into the form needed. At this time, it is necessary to shuffle the expressions in the numerator and focus on the expressions in the denominator. Now, express the expression for the exponential position in factored form. Now express the expression in the denominator as a power of the exponential function using the power-of-exponent rule. Exponents of the expressions in both numerator and denominator are the same, and it can be simplified by the force of the quotient rule. The limit of the base rational expression can be obtained by the limit quotient rule. It helps people evaluate limit of any rational expression by taking the quotient of the limit of the expression.

The fifth question is

$$L = \lim_{y \rightarrow 0^+} \cos(2y)^{\frac{1}{y^2}} \quad (12)$$

First of all, one can do some manipulation on this to turn it into a problem that can be done with L'Hospital's rule. By defining $z = \cos(2y)^{\frac{1}{y^2}}$ and take the log of both sides, it is found that

$$\ln z = \ln \cos(2y)^{\frac{1}{y^2}} = \frac{1}{y^2} \ln \cos(2y)^{\frac{1}{y^2}} \quad (13)$$

Next, before proceeding, it is necessary to notice that the following relation, namely, as $y \rightarrow 0^+$, $\ln \cos(2y)^{\frac{1}{y^2}} \rightarrow \frac{\ln 1}{0} = \frac{0}{0}$. By virtue of the L'Hospital's rule, it is found that $\lim_{y \rightarrow 0^+} \ln z = 2$. Recalling that $z = e^{\ln z}$, the original limit is e^{-2} .

4. Conclusion

To sum up, in the process of solving limit problems, L'Hopital's rule plays an extremely important role, and the above method is an important method to find the limit in advanced mathematics. When doing the problem of solving the limit, it is not enough to just master the above methods without being able to thoroughly and clearly understand the conditions required for the above methods. Careful analysis and careful selection are necessary. There is more than one method used for a question of seeking the limit. Everyone can integrate and choose an appropriate method. This will not only increase the accuracy, but also save a lot of unnecessary troubles and achieve twice the result with half the effort. As a student, the author has a limited level of knowledge and understanding. This article only discusses the most common limit calculation methods, and the examples given are relatively simple. I hope these calculation methods will be helpful to students who are learning limit. When using L'Hopital's rule, there are several points that are needed to pay attention to. Firstly, theoretical conditions must be satisfied when using L'Hopital's rule. This means that both numerator and denominator either tend to zero or to infinity simultaneously. Secondly, L'Hopital's rule should be simplified before use and after operation. For ease of calculation, when using L'Hopital's rule, one can combine advanced simplifications such as equivalent substitution. Lastly, L'Hopital's rule is only a tool for solving representative problems of limit rather than a panacea, and it is sometimes should be combined with other methods.

References

- [1] Stein E. M. and Rami Shakarchi. (2009). Real Analysis. Princeton University Press.
- [2] Abramowitz M. and Stegun I. A. (1972). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover.
- [3] Xu H. (2016). The limit problem of integration from two different points of view, studies in college mathematics, 19(01): 71-74.
- [4] Zhao Lili. (2022). Common Problem Types and Problem – solving Ideas About Basic Concepts and Properties of Definite Integrals. Journal of Henan Institute of Education (Natural Science Edition), 31(4): 44-48.
- [5] Zheng Baojie and Duan Yuxuan. (2021). Techniques for Solving Definite Integrals of a Class of Trigonometric Functions. Journal of Henan Institute of Education (Natural Science Edition), 30(4): 51-54.
- [6] Zhao X. and Yang L. (2022). Analysis of function limit based on Matlab software. Chia Academic Journal Electronic Publishing House, 51(11): 121-123.
- [7] Xiao H. and Zhang Y. (2022). Research on limit concept teaching based on symbolic cognition. Journal of Hubei Industrial Polutechnic, 35(05): 81-84.
- [8] Wang P., Xie S., Xu Q., and Zhang Z. (2022). Multiple methods and visualization of the limit of a power exponential composite function. Journal of Hengyang Normal University (Natural Science), 43(06): 19-21.
- [9] Xu F. and Fan J. (2022). On limit problems related to integrals. Studies in College Mathematics, 25(02): 60-62.
- [10] Zhu J.-M. and Luo Q.-M. (2021). A novel proof of two partial fraction decompositions. Advances in Difference Equations, 2021: 274.