

The Contour Integral Approach to Several Improper Integrals with Hyperbolic Functions

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Abstract. In the field of Complex Analysis, it is acknowledged that Cauchy's Residue Theorem plays an essential role, which allows the calculation of complex integrals by adding up the residues of singularities in the complex plane. Many mathematicians have developed various theorems out of Cauchy's Residue Theorem and have solved numerous problems using Cauchy's Residue Theorem, but there are still a lot more studies needed. Thus, this paper focuses on examining Residue Theorem deeply by introducing singularity point and residue, combining Laurent series and complex integral, then deducting Cauchy's Residue Theorem. This paper then concentrates on solving four unique complex integrals to illustrate Cauchy's Residue Theorem by analyzing the graph of integration, reformatting the integrals, applying theorems or tricks, integrating the reformatted integrals, and simplifying the results. As a result, this paper not only presented a deeper analysis of the deduction of Cauchy's Residue Theorem, but also presented the solutions towards four previously unsolved complex integrals. The deduction of Cauchy's Residue Theorem and the four complex analysis problems have important applications for dealing with integrals associated with hyperbolic functions and lead to future research in other areas of mathematics and physics.

Keywords: Cauchy's Residue Theorem; complex integral; isolated singularity point; singularities; residue; complex number.

1. Introduction

Calculus is one of the most important parts in Mathematics and it has various applications in many other fields. For example, in mechanics, calculus is widely used in kinematics to find the velocity of a particle by integrating acceleration either with respect to time or displacement and thus put velocity into the integral to calculate displacement, and in physics, integration can help calculate the rate of change. Regarding to the complex variables, different from the function that takes real numbers as variables, it is a function that takes complex numbers with both real part and imaginary part as independent and dependent variables [1].

Complex variable functions have a history of nearly three hundred years, it was born in the 18th century when Euler mentioned two equations derived from the integration of complex functions in one of his dissertations [2]. Even earlier than him, French mathematician Darren Bell got these functions in his paper about fluid mechanics. The further advancement of complex variable theory took place in the 19th century. During that period of time, mathematicians regarded complex function theory as the largest subfield of mathematics. Euler and D'Alembert did the earliest work in the creation of complex variable theory, and Laplace later studied the integration of complex variable function [3]. Then, Cauchy, Riemann and German mathematician Weierstrass have done a lot of foundation work for the development of this discipline. At the beginning of the 20th century, the students of Weierstrass, Swedish mathematician Leffler, French mathematician Poincare, Adama, etc. have done a lot of research work, paving the way for further research of complex functions and opening up a broader research field in many disciplines. One specific example of the use of complex

function is airplane design, the Russian scientist Dhakovsky solved the structural problem of an airplane wing by using the theory of complex functions [4]. He also studied partial differential equations and their approximate integration methods, these were widely applied to aerodynamics and fluid mechanics.

2. The Residue Theorem

Cauchy's Residue formulae is set based on the Cauchy's integration formulae which states that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz, \quad (1)$$

Where function f is defined analytic everywhere inside and on a contour C which is simple and closed and z_0 is any point lies inside C . In order to describe the Cauchy's residue formula, the article have to introduce the topic of isolated singular points. A function f is called analytic at a point z_0 if it is differentiable at each point in some neighborhood of z_0 . By contrast, if f is not analytic at z_0 but satisfies the fact that it is analytic at some point in every neighborhood of it, then the point z_0 is defined as a singular point of f . The Cauchy's residue theorem concentrates on a particular type of singularity point which is named isolated singularity point. The singular point z_0 is able to be regarded isolated if there is a deleted ε neighborhood $0 < |z - z_0| < \varepsilon$ of z_0 throughout which f is analytic. In other words, the isolated points can be regarded as the points which make the complex variable have no sense [5].

When z_0 is an isolated singular point of a function f , which makes the function does not exist, then the complex plane exists a number R_2 which is larger than zero such that f has the characteristic of analytic at each single point z for which $0 < |z - z_0| < R_2$. So $f(z)$ can be written into the following Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_n}{(z-z_0)^n} + \dots \quad (2)$$

Where $0 < |z - z_0| < R_2$ [6]. The coefficients a_n and b_n can be represented in the integral form

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{(z-z_0)^{n+1}} \quad (n = 1, 2, \dots) \quad (3)$$

Where C represents a simple closed contour that encloses the isolated singular point z_0 that lies in the punctured ε neighborhood. In the case $n = 1$, b_1 can be expressed into the form $b_1 = \frac{1}{2\pi i} \int_C f(z) dz$. Therefore, $\int_C f(z) dz = 2\pi i b_1$. If C is a contour with the feature of simple and closed, taking the positive direction and make the assumption that a function f is analytic inside and on C , excluding a finite number of singular points z_k ($k = 1, 2, 3 \dots n$) which lie in the contour C , then [7]

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}[f(z), z = z_k] \quad (4)$$

3. Examples and Solutions

3.1. Hyperbolic Sine Integral

The first example is [8]

$$I = \int_0^{\infty} \frac{x}{\sinh x} dx. \quad (5)$$

Notice that $\sinh x = \frac{e^x - e^{-x}}{2}$ and assume $u = e^x$, so $x = \ln u$, $dx = (\ln u)' du = \frac{1}{u} du$. Therefore, the integral in Eq. (5) is

$$I = \int_0^{\infty} \frac{x}{\sinh x} dx = 2 \int_1^0 \frac{\ln u}{u^2-1} du. \quad (6)$$

Using the method of integration by substitution again and assuming $u = \frac{1}{t}$, hence $du = \left(\frac{1}{t}\right)' dt = -\frac{1}{t^2} dt$. Substituting these into Eq. (6), it is found that

$$I = 2 \int_1^0 -\ln t \cdot \frac{1}{t^2-1} \left(-\frac{1}{t^2}\right) dt = 2 \int_1^0 \ln t \cdot \frac{1}{1-t^2} dt. \quad (7)$$

Here, $\frac{1}{1-t^2}$ is a infinity series with first term 1 and common ratio t^2 , i.e., $\frac{1}{1-t^2} = 1 + t^2 + t^4 + \dots$. By substituting,

$$I = 2 \int_1^0 \ln t \cdot (1 + t^2 + t^4 + \dots) dt = 2 \sum_{k=0}^{\infty} \int_1^0 \ln t \cdot t^{2k} dt. \quad (8)$$

Next, using the method of integration by parts $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$ and let $u = \ln t$, $\frac{dv}{dt} = t^{2k}$. Hence, $\frac{du}{dt} = \frac{1}{t}$, $v = \frac{t^{2k+1}}{2k+1}$. Substituting these into the function,

$$\int_1^0 \ln t \cdot t^{2k} dt = \ln t \cdot \frac{t^{2k+1}}{2k+1} - \int_1^0 \frac{t^{2k+1}}{2k+1} \frac{1}{t} dt = \left(\frac{1}{2k+1}\right)^2. \quad (9)$$

Hence,

$$\int_0^{\infty} \frac{x}{\sinh x} dx = 2 \sum_{k=0}^{\infty} \left(\frac{1}{2k+1}\right)^2 = 2 \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right) - 2 \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \dots\right). \quad (10)$$

By using of the identity $\sum_{k=0}^{\infty} \frac{1}{n^2} = \lim_{n \rightarrow \infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2}\right)$, it is immediately to find that

$$\int_0^{\infty} \frac{x}{\sinh x} dx = \frac{\pi^2}{3} - \frac{\pi^2}{12} = \frac{\pi^2}{4}. \quad (11)$$

3.2. Hyperbolic Cosine Integral

The second example is [9]

$$I(a) = \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh(x)} dx \quad (12)$$

where $-1 < R(a) < 1$. This integral can be calculated by using of residue calculus with the help of a rectangular contour. Under this circumstance the question requires people to attempt to integrate the real part of the complex variable. To simplify this problem, the first process is to consider the integral $I(a) = \oint_C \frac{e^{az}}{\cosh(z)} dz$. Subsequently, since the question is required to apply Cauchy's Residue Theorem, the first step is to find the isolated singular point z_0 . So the purpose of this step is to find the value of z at which the function does not exist.

Since e^{az} is a non-zero constant, the value of z such that $\cosh(z)$ equals to zero is required. According to the basic conversion between hyperbolic function and trigonometric function, the following equation can be easily written $\cosh(z) = \cos(iz) = 0$, which gives the value of the isolated singular point of this function $z = \frac{i\pi}{2} - k\pi i, k \in \mathbb{Z}$.

Let $L > 0$ be a real number, and C_1, C_2, C_3, C_4 be the line segments that go from $-L$ to L , from L to $L + \pi i$, from $L + \pi i$ to $-L + \pi i$, from $-L + \pi i$ to $-L$. Thus a rectangular contour is constructed. Using the previously constructed rectangular contour with center $\left(0, \frac{i\pi}{2}\right)$, the integral along the contour can be rewritten as:

$$I(a) = \int_{-L}^L \frac{e^{ax}}{\cosh(x)} dx + \int_0^{\pi} \frac{e^{a(L+iy)} dy}{\cosh(L+iy)} + \int_L^{-L} \frac{e^{a(x+\pi i)} dx}{\cosh(x+\pi i)} + \int_{\pi}^0 \frac{e^{a(-L+iy)} dy}{\cosh(-L+iy)}. \quad (13)$$

Now, by applying Residue Theorem, the value of the integral of a complex variable inside a contour is given by $2\pi i$ multiply the residue(s) at the isolated singularity point(s) inside the contour. Therefore,

$$I(a) = 2\pi i \operatorname{Res}\left(\frac{e^{az}}{\cosh(z)}; \frac{i\pi}{2}\right) = 2\pi i \lim_{z \rightarrow \frac{i\pi}{2}} \frac{e^{az}\left(z - \frac{i\pi}{2}\right)}{\cos(iz)}. \quad (14)$$

By L'Hospital's Rule, the limit can be calculated as:

$$2\pi i \lim_{z \rightarrow \frac{i\pi}{2}} \frac{e^{az}\left(z - \frac{i\pi}{2}\right)}{\cos(iz)} = 2\pi i \lim_{z \rightarrow \frac{i\pi}{2}} \frac{ae^{az}z + e^{az} - ae^{az}\frac{i\pi}{2}}{-i\sin(iz)} = 2\pi e^{a\frac{i\pi}{2}}. \quad (15)$$

Joining together, $\oint_C \frac{e^{az}dz}{\cosh(z)} = 2\pi e^{a\frac{i\pi}{2}}$. Additionally, since $\cosh(x + \pi i) = -\cosh(x)$, the two components of the integral can be combined into the form: $\int_{-L}^L \frac{e^{ax}dx}{\cosh(x)} + \int_L^{-L} \frac{e^{a(x+\pi i)}dx}{\cosh(x+\pi i)} = (1 + e^{a\pi i}) \int_{-L}^L \frac{e^{ax}dx}{\cosh(x)}$

As $L \rightarrow \infty$, the rest of the components of the integrals $\int_0^\pi \frac{e^{a(L+iy)}dy}{\cosh(L+iy)}$ and $\int_\pi^0 \frac{e^{a(-L+iy)}dy}{\cosh(-L+iy)} \rightarrow 0$. In other words, as L tends to infinity, the vertical integrals is eliminated. This is because

$$\left| \int_0^\pi \frac{e^{a(L+iy)}dy}{\cosh(L+iy)} \right| \leq \int_0^\pi \left| \frac{2e^{aL}}{e^L e^{iy} + e^{-L}} \right| dy \rightarrow \int_0^\pi |2e^{(a-1)L}| dy \rightarrow 0 \quad (16)$$

As $L \rightarrow \infty$. Thus, $2\pi e^{a\frac{i\pi}{2}} = (1 + e^{a\pi i}) \int_{-L}^L \frac{e^{ax}dx}{\cosh(x)}$. Taken together,

$$\int_{-\infty}^\infty \frac{e^{ax}dx}{\cosh(x)} = \frac{2\pi e^{a\frac{i\pi}{2}}}{1 + e^{a\pi i}} = \frac{\pi}{\cos\left(\frac{a}{2}\pi\right)}. \quad (17)$$

3.3. Double Hyperbolic Sine/Cosine Integrals

The third example is [10]

$$I(a, b) = \int_{-\infty}^\infty \frac{\sinh(ax)}{\sinh(bx)} dx \quad (18)$$

with $a < b$. First, turning it into $\frac{\sinh(ax)}{\sinh(x)}$ and integrating over the real line, where $a \in (-1, 1)$. Since this is an even function, only $\int_0^\infty \frac{\sinh(ax)}{\sinh(x)} dx$ needs to be evaluated and it follows that

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(x)} dx = \int_0^\infty \frac{e^{ax} - e^{-ax}}{e^x - e^{-x}} dx = \int_0^\infty \frac{e^{ax-x} - e^{-ax-x}}{1 - e^{-2x}} dx. \quad (19)$$

Taking the series expansion of $\frac{1}{1 - e^{-2x}}$, then

$$\int_0^\infty \frac{\sinh(ax)}{\sinh(x)} dx = \sum_{m \geq 0} \left[\frac{1}{2m+1-a} - \frac{1}{a+1-2m} \right] = \sum_{m \geq 0} \frac{2a}{(2m+1)^2 - a^2} = \frac{\pi}{2} \tan \frac{\pi a}{2}. \quad (20)$$

Thus, $\int_{-\infty}^\infty \frac{\sinh(ax)}{\sinh(bx)} dx = \frac{\pi}{b} \tan\left(\frac{\pi a}{2b}\right)$.

Actually, one can also calculate a related integral

$$I(a) = \int_0^\infty \frac{\cosh(ax)}{\cosh(\pi x)} dx = \frac{1}{2 \cos(a/2)}. \quad (21)$$

The idea is quite similar and the procedure is omitted.

4. Conclusion

In conclusion, the Cauchy's residue theorem plays a vital role in mathematics, especially in the application of integrals related to hyperbolic functions, and it can be used to calculate infinite series of real integrals which are complicated to compute using the classical method of integration. The Residue Theorem is mainly the inference of Cauchy's integral theorem and Cauchy's integration formula. In this paper, the residue theorem and Cauchy integral formula are mainly used, and the goal of this work is to thoroughly examine Cauchy's residue theorem by introducing singularity point and residue, merging Laurent series and complex integral, and deducing it. The examples considered in this paper are hyperbolic sine integral, hyperbolic cosine integral, and comparison of hyperbolic sine/cosine integrals. It is interesting to note that results of these integrals relate to the famous constant π . Beyond the theoretical aspect, the complex variable functions are an extremely important existence in modern physics, such as solving linear equations in circuit theory, wave functions in quantum mechanics, two-word positions, where Wick's rotation recognizes "i", and many-body theory integral application.

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