

Unveiling Several Intricate Integrals with Inverse Tangent Function

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Abstract. Calculus is the study of successive transformations in mathematics. Calculus includes two parts: differentiation and integration. Using the basic theorem of calculus, differentiation can be transformed into integration. Calculus use the infinite sequence and infinite series to converge to define the basic definition of limits. Differentiation can be simply understood as instantaneous change or just the slope of a function. The integration can be simply defined as calculate the area under the function. Integrals are divided into definite integrals and indefinite integrals. In the following, the paper mainly studies some specific definite integrals which related to inverse tangent function, polynomial, and logarithmic function. Through the method of substitution, integration by parts, exchange of integral order, and triangular substitution, all the answers have a rule, which is the integral related to π . Therefore, it is interesting to unveil the possible general solution of these integrals.

Keywords: substitution; integration by parts; exchange of integral order; triangular substitution.

1. Introduction

In math, calculus is mainly talking about the basic mathematical analysis, which are mainly discussing functions and limits [1]. Calculus can divide in Modern and original. In fact, calculus can be divided into two parts through the timeline, new and old. Of course, what people generally learn now is new. Calculus was the first achievement of modern mathematics. It's the greatest and biggest process of the logical in mathematical analysis. Calculus is used in so many subjects, especially in Physics [2]. In classical mechanics and electromagnetism all the concepts require calculus for answers. Or in the motion of an object, using calculus can find its path from knowing the acceleration of an object. In economics, there is an easy way by calculus to Calculate Marginal Cost and Marginal Revenue [3]. In medicine, the best branching angle of a blood vessel to maximize flow can be funded by calculus. For example, in optics, the propagation of light waves in the near-field region can be calculated by the Fresnel diffraction integral [4]. For the Fresnel diffraction integral, assuming that the aperture size is much smaller than the propagation path length, then $K(x) \approx 1$, especially in small areas around the z-axis. If $x, y \ll z$, denominator $R \approx Z$, now it uses Fresnel approximation:

$$\Psi(x, y, z) = -\frac{ie^{ikz}}{\lambda Z} \int_S \Psi(x', y', 0) e^{\frac{ik[(x-x')^2 + (y-y')^2]}{2x}} dx' dy'. \quad (1)$$

The most basic arithmetic rules of definite integral are substitution, integration by parts, triangular substitution, exchange of integral order, tangent half angle substitution and so on [5].

Substitution can be defined as: suppose $f(x)$ is Integrable function, $g = g(x)$ is a continuously differentiable function [6]. It follows that

$$\int_{\alpha}^{\beta} f(g)g'dx = \int_{g(\alpha)}^{g(\beta)} f(g)dg. \quad (2)$$

Integration by parts can be defined as follows. In calculus, an equation is written by using a relationship between the derivative of one part and the antiderivative of another part equal to the original equation. This is called integration by parts, i.e.,

$$\int u dv = uv - \int v du. \quad (3)$$

Triangle exchange is actually a special exchange method, the most basic publicity is [7]

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c, \quad (4)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c, \quad (5)$$

$$\int \frac{dx}{x^2-a^2} = a^2 \int ((\sec \theta)^3 - \sec \theta) d\theta. \quad (6)$$

Tangent half angle substitution: The half-width formula is actually a special substitution formula, through which the trigonometric functions of x are replaced with ordinary equations of t by setting $t = \tan \frac{x}{2}$. Its general formula is:

$$\int f(\sin x, \cos x) dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2dt}{1+t^2}. \quad (7)$$

2. Examples with Solutions

2.1. Example I

Compute the integral [8]

$$I = \int_0^\infty \frac{\tan^{-1}(x^2)}{x^4+x^2+1} dx. \quad (8)$$

By using of the identity $\tan^{-1}(t) + \tan^{-1}(1/t) = \frac{\pi}{2}$, the integral in Eq. (8) can be rewritten as

$$2I = \frac{\pi}{2} \int_0^\infty \frac{x^2}{x^4+x^2+1} dx + \int_0^\infty \frac{(1-x^2) \tan^{-1}(x^2)}{x^4+x^2+1} dx \quad (9)$$

After some easy calculation, it follows that

$$Z = \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{2\sqrt{3}} - \frac{1}{2} \int_0^\infty \frac{(x^2-1) \tan^{-1}(x^2)}{x^4+x^2+1} dx = \frac{\pi^2}{8\sqrt{3}} - \frac{1}{2} J \quad (10)$$

Where

$$J = \frac{(x^2-1) \tan^{-1}(x^2)}{x^4+x^2+1} dx. \quad (11)$$

Now, it only needs to calculate J , with a very clever operation to make the derivative of the equation equal to the original equation.

$$J = \int_0^\infty \tan^{-1}(x^2) \cdot \left(\frac{1}{2} \ln \left(\frac{x^2-x+1}{x^2+x+1}\right)\right)' dx. \quad (12)$$

And by integrating by parts:

$$J = \int_0^\infty \frac{x}{1+x^2} \ln \left(\frac{x^2+x+1}{x^2-x+1}\right) dx \quad (13)$$

Next, through the substitution of $x = \tan(c)$ and the periodicity of trigonometric functions:

$$J = \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{1+(\sin x)^2} \ln \left(\frac{2+\cos(x)}{2-\cos(x)}\right) dx \quad (14)$$

Then through tangent half-angle substitution:

$$J = \int_0^\infty \frac{1-c^2}{c^4+4c^2+1} \ln \left(\frac{c^2+1}{3c^2+1}\right) dc \quad (15)$$

Calculate the numerator and denominator separately.

$$J = 2 \int_0^\infty \frac{1-x^2}{x^4+4x^2+1} \ln \left(\frac{x^2+3}{x}\right) dx \quad (16)$$

Next use that clever operation again,

$$J = \int_0^\infty \tan^{-1}\left(\frac{2x}{1+x^2}\right) \frac{dx}{x} - \int_0^\infty \tan^{-1}\left(\frac{2x}{1+x^2}\right) \left(\frac{2x}{3+x^2}\right) dx = J1 + J2 \quad (17)$$

Then calculating both functions separately, from tangent half-angle substitution, it follows that:

$$J1 = 2 \int_0^{\frac{\pi}{2}} \frac{\tan^{-1}(\sin x)}{\sin x} dx \quad (18)$$

$$\frac{\tan^{-1}(\sin x)}{\sin x} = \int_0^1 \frac{dy}{1+(\sin x^2)y^2} \quad (19)$$

$$J1 = 2 \int_0^{\frac{\pi}{2}} \int_0^1 \frac{dy}{1+(\sin x^2)y^2} dx \quad (20)$$

From Swap order of integration and triangular substitution method, it follows that

$$J1 = \int_0^1 \int_0^{\frac{\pi}{2}} \frac{1}{1+(\sin^2 x)y^2} dx dy = \pi \ln(1 + \sqrt{2}) \quad (21)$$

$$J2 = \frac{\pi}{2} \ln(2 + \sqrt{3}) \quad (22)$$

$$J = \pi \ln(1 + \sqrt{2}) - \frac{\pi}{2} \ln(2 + \sqrt{3}) \quad (23)$$

$$Z = \frac{\pi^2}{8\sqrt{3}} + \frac{\pi}{4} \ln(2 + \sqrt{3}) - \frac{1}{2} \pi \ln(1 + \sqrt{2}) \quad (24)$$

2.2. Example II

Compute the integral [9]

$$\int_{-\infty}^{\infty} \frac{x \tan^{-1}\left(\frac{1}{x}\right) \log(1+x^2)}{1+x^2} dx \quad (25)$$

After integration by parts, the function can be written as

$$\int_0^{+\infty} \frac{(\log(1+x^2))^2}{1+x^2} dx. \quad (26)$$

Then substitute by $x = \frac{1}{x}$ and limit the interval to $x = 1$,

$$\int_0^1 \frac{(\log(1+x^2))^2 + (\log 1 + \frac{1}{x^2})^2}{1+x^2} dx \quad (27)$$

Substituting by $x = \frac{2x}{x^2+1}$ and $x = \sqrt{1-x^2}$, the function is:

$$J = (\log 2)^2 \int_0^1 \frac{dx}{\sqrt{1-x^2}} - (\log 2)^2 \int_0^1 \frac{\log(1-x^2)}{\sqrt{1-x^2}} dx + \frac{1}{2} \int_0^1 \frac{(\log(1-x))^2 + (\log(1+x))^2}{\sqrt{1-x^2}} dx \quad (28)$$

According $(a+b)(a-b) = a^2 - b^2$, $(a \pm b)^2 = a^2 + b^2 \pm 2ab$, it is found that

$$J = (\log 2)^2 \int_0^1 \frac{dx}{\sqrt{1-x^2}} - \log(2) \int_0^1 \frac{\log(1-x^2)}{\sqrt{1-x^2}} dx + \frac{1}{2} \int_0^1 \frac{(\log(1-x))^2}{\sqrt{1-x^2}} dx - \int_0^1 \frac{\log(1+x) \log(1-x)}{\sqrt{1-x^2}} dx \quad (29)$$

Then substituting by $x = \sin^{-1}(x)$, it follows that

$$J = (\log 2)^2 \int_0^{\frac{\pi}{2}} dx - 2 \log(2) \int_0^{\frac{\pi}{2}} \log(\cos(x)) dx + 2 \int_0^{\frac{\pi}{2}} (\log(\cos x))^2 dx - \int_0^{\frac{\pi}{2}} \log(1 + \sin(x)) \log(1 - \sin(x)) dx = 2\pi (\log(2))^2 + \frac{\pi^3}{6}. \quad (30)$$

2.3. Example III

Compute the integral [10]

$$Z = \int_0^\infty \frac{(1-x^2) \tan^{-1}(x^2)}{x^4 + 4x^2 + 1} dx \quad (31)$$

First, consider another function first,

$$J(q) = \int_0^\infty \frac{\tan^{-1}(qx^2)}{1+x^2} dx. \quad (32)$$

Express Eq. (32) in another form:

$$J(q) = \int_0^\infty \int_0^y \frac{x^2}{(1+t^2x^4)(1+x^2)} dt dx. \quad (33)$$

From Swap order of integration and simplifying the form, (33) can be written as:

$$J(q) = \int_0^y dt \int_0^\infty \frac{1+tx^2}{(1+x^4)(1+t^2)\sqrt{t}} dx - \int_0^y dt \int_0^\infty \frac{1}{(1+t^2)(1+x^2)} dx \quad (34)$$

Perform certain calculations on the infinite integral first:

$$\begin{aligned} J(q) &= \frac{\sqrt{2\pi}}{4} \int_0^y \frac{1+t}{(1+t^2)\sqrt{t}} dt - \frac{\pi}{2} \int_0^y \frac{1}{1+t^2} dt \\ &= \frac{\pi}{2} [\tan^{-1}(\sqrt{2y+1}) + \tan^{-1}(\sqrt{2y-1}) - \tan^{-1} y] \end{aligned} \quad (35)$$

As it known, z can be written as

$$z = \int_0^\infty \frac{(\sqrt{3}-1) \tan^{-1}(x^2)}{2(x^2+2-\sqrt{3})} dx - \int_0^\infty \frac{(\sqrt{3}+1) \tan^{-1}(x^2)}{2(x^2+2+\sqrt{3})} dx \quad (36)$$

It follows that

$$\begin{aligned} z &= \frac{-\sqrt{2}\pi}{4} [\tan^{-1}(\sqrt{2y+1}) + \tan^{-1}(\sqrt{2y-1}) - \tan^{-1} y] \Big|_{\frac{2+\sqrt{3}}{2-\sqrt{3}}} \\ &= \frac{-\sqrt{2}\pi}{4} [-\tan^{-1}(\sqrt{3}-2) + \tan^{-1}(2-\sqrt{3})] = -\frac{\sqrt{2}\pi^2}{24}. \end{aligned} \quad (37)$$

2.4. Example IV

Compute the integral

$$g(x) = \int_0^\infty Li_2(e^{-\pi x}) \tan^{-1} x dx. \quad (38)$$

Integrate by parts first:

$$g(x) = 0 + \int_0^\infty \frac{Li_3(e^{-\pi x})}{\pi(1+x^2)} dx \quad (39)$$

Then one can simplify it

$$g(x) = \frac{\pi^2}{2} \int_0^\infty \left(\frac{1}{x^4} + \frac{1}{3x^2} - \frac{\coth x}{x^3} \right) e^{-x} dx. \quad (40)$$

In order to evaluate the last integral, let us introduce the function $I(s)$ defined by

$$I(s) = \frac{\pi^2}{2} \int_0^\infty \left(\frac{1}{x^4} + \frac{1}{3x^2} - \frac{\coth x}{x^3} \right) x^s e^{-x} dx \quad (41)$$

Then $I(s)$ is analytic for $R(s) > 1$ and our integral can be written as $I(0)$. Now assume for a moment that $R(x) > 3$, then Eq. (41) can be continued as

$$I(s) = \frac{\pi^2}{2} \int_0^\infty \left(\frac{1}{x^4} + \frac{1}{x^3} + \frac{1}{3x^2} - \frac{2}{x^3(1-e^{-2x})} \right) x^s e^{-x} dx = \frac{\pi^2}{2} (\Gamma(s-3) + \Gamma(s-2) + \frac{1}{3}\Gamma(s-1) - 2\Gamma(s-2)(1-2^{2-s})\zeta(s-2)) \quad (42)$$

Using the functional equation of $\zeta(s)$ and the reflection formula for $\Gamma(s)$

$$2\Gamma(s-2)(1-2^{2-s})\zeta(s-2) = \frac{(1-2^{s-2})\pi^{s-2}}{\cos(s\pi/2)}\zeta(3-s) \quad (43)$$

Taking $s \rightarrow 0$, this converges to $\frac{3}{4\pi^2}\zeta(3)$. Combining these two observations, it follows that

$$I(0) = \frac{\pi^2}{18} - \frac{3}{8}\zeta(3). \quad (44)$$

3. Conclusion

To conclude, substitution, integration by parts, triangular substitution, exchange of integral order, and tangent half angle substitution are simplification ways to compute definite integrals. The answer to each selected equation can be related to π in the end, but the methods that need to be used for each question are different. In example I, all the formulas mentioned above have been used, and a lot of work is required. Only experience can get the answer. In fact, some formulas are also used in example II, but the solution to the problem is much simpler and more direct than the previous problem, and there are fewer steps. Is it possible to solve this kind of definite integral through these steps? In fact, example III is different. It's not as difficult as the previous two questions, but the idea is very important. Quoting another integral to get a general answer is needed when solving example III and example IV are very cunning. It went smoothly at the beginning, but in the end, it still has to use an uncommon theorem and conversion of formulas to get the answer. In fact, there is no perfect formula for these types of equations. However, as long as people solve them step by step as normal problems, the simple and direct answers can be obtained ultimately.

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