

Using Differential Equation to Explain the Simple Harmonic Motion Equation

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Abstract. It aims to explain the motion law of simple harmonic vibration in phase space with the complex part of differential equation. Introduce and enumerate differential equations, classify and discuss each case, and finally get the appropriate formula. Angular velocity, phase, displacement, time, amplitude, acceleration and trigonometric function are connected by equation solution. It explains the rationality of physics in phase space, and does not restrict the limited operation with real space. Using the knowledge of second order differential equation, the formula of angular velocity of simple harmonic motion is successfully proved. According to the equation of motion, Displacement is a cosine function with respect to time t ; The velocity is a sine function with respect to time t ; and the acceleration is a sine function with respect to time t of two angular velocity multiples. The emergence of differential equations is equivalent to opening up a new way for the scientific community, so many problems in other disciplines need to be solved and proved. Therefore, it is a very reasonable behavior to combine physics with this method.

Keywords: Differential Equation, Harmonic Motion, Cosine Function.

1. Introduction

In classical mechanics, with the change of time value, the motion of an object is described by its position and velocity. Newton's law allows these variables to be dynamically expressed as differential equations that vary with time as unknown positions of the body; In some cases, this differential equation can be solved explicitly[1].

At first, mathematicians focused on finding the general solution of differential equations. Later, it proved that it was generally impossible, so they gradually gave up this extravagant hope and turned to the definite solution problems: initial value problems, boundary value problems, mixed problems, etc. However, even for first-order ordinary differential equations, the elementary solution has also been proved impossible, turning to numerical methods and numerical methods is available, which first need to solve the theoretical problems of the existence and uniqueness of the solution[2].

In the 1970s, with the infiltration of mathematics into chemistry and biology, a large number of reaction-diffusion equations appeared.

The solution of the equation contains any element to make as many changes as possible, people can get all the solutions of the equation, so mathematicians call this solution containing any element "general solution".

For a long time, people have devoted themselves to "seeking a general solution". But the following three reasons make this effort of "seeking a general solution" gradually abandoned.

First, there are obviously few equations that can get a general solution. In terms of ordinary differential equations, the number of general solutions of first-order equations is very small, except for linear equations, separable variable equations and equations that are transformed into these two equations by special methods. If the general solution is regarded as a certain kind of inverse operation of calculus and elimination, it is tentative and has no certain rules, even sometimes impossible, just like the well-known inverse operation. Moreover, this general solution also increases the difficulty of solution with the increase of its degree of freedom.

Second, when people want to clarify the meaning of general interpretation, they will encounter serious ambiguity. This mainly occurs in the study of partial differential equations.

Third, the important application of differential equations in physics and mechanics is not to find any solution of the equation, but to find a solution that meets some supplementary conditions. Cauchy believes that this is the most important and decisive reason for giving up "seeking a general solution". These supplementary conditions are the definite solution conditions. Finding the solution of the equation that meets the conditions of definite solution is called solving the problem of definite solution.

2. Basic concepts

Differential equations are equations with unknown differential terms.

The equation with first derivative is a first-order differential equation.

The equation with second derivative is a second-order differential equation.

The concept of derivative, differential and rate of change need to be involved.

Then, the derivative expression of the function represents the effect of its domain value and the change rate of the corresponding point of the original data

The next work is to expand this concept.

If an equation contains elements of differential expression, then the equation is called "differential equation"

Similar to the concept of derivative, differential equations are divided into two general branches: first order and second order:

The purpose and significance of studying differential equations.

The differential equation is divided into three levels, and it is good to advance layer by layer.

Master the solutions of six common equations, and see that an equation can identify its type and solve it correctly.

Combined with other knowledge points, such as the mean value theorem, which can be done with differential equations, is called universal construction method, which is very useful for proving the equality of linear differential equation structure. And the structure of the solution of the second-order variable coefficient differential equation, which is actually the content of the line generation part. As well as differential equations and variable limit integrals, functions can be found.

3. The application of differential equations, geometric applications and physical applications, this is actually the most difficult, because it needs to list differential equations, but ordinary people's modeling ability is very low, it is very difficult to list or make errors, and the determination of constants in the result of solution also needs to be determined by many implicit relationships. For example, if a piecewise function has a piecewise point of $(0, 0)$ and the function is continuous and smooth in all definition fields, nothing can be seen.

4. Euler equations and differential equations of equations.

Phase space:

The mechanical equation of classical mechanical system is the second derivative of position. There is no need to know the third derivative of displacement. Of course, some equations of motion in general relativity depend on higher derivative. Combined with the theory of differential equations, we must know the initial displacement and initial velocity of physics in order to predict the system. However, solving second-order differential equations is very annoying, so physicists convert a second-order differential equation into two first-order differential equations through the following definitions. The initial conditions needed to solve the differential equations are the initial displacement and initial velocity, and the equations are effectively simplified[3].

The trajectory of a particle with known motion should be calibrated by six parameters. The open space of these parameters is a kind of phase space.

Mechanical quantities are often the combination of generalized coordinates and generalized momentum, which can define a phase space, in which the independent variables are generalized coordinates and generalized momentum respectively, and its cross section, that is, taking several coordinates or their linear combination.

Therefore, phase space is a very basic physical fact in theoretical mechanics.

If people want to understand phase space correctly, they must start with calculus, because phase space is essentially the geometric properties of the equation found in the process of solving the equation.

The starting point of space is this second-order differential equation, in which the velocity and displacement are regarded as two different variables, so that the original n second-order ordinary differential equations with n unknowns are transformed into $2n$ first-order ordinary differential equations with $2n$ unknowns to study.

At first glance, why should do this? With more unknowns and more equations, what's the significance?

The theory of ordinary differential equations is important, differential geometry, Lie groups and Lie algebras, know that for the first-order ordinary differential equations, mathematicians have long developed a set of particularly complete theories on the existence, uniqueness, evolution law, geometric properties, stability and so on. Therefore, if the problem of dynamics is transformed into a problem in phase space, people can know the important properties of this dynamic system, such as conservation, stability and integrability, without specifically solving the equation. Solving the result without numerical method? When the degree of freedom of the system is particularly large, if the most primitive method is used to calculate these properties, even if the result can be calculated, it is difficult to detect what the system is invariant, because many times, the specific law of a system is not the most important, but what the conserved quantity of the system is, because the conserved quantity is often directly observable, which means that it can be used to characterize the state of the system[4].

In summary, what people most want to get is the analytical solution, which can be regressed to obtain the numerical solution. But when the problem is very complex, even if it get the above things, it is difficult to summarize the important properties behind the chaotic and complex motion law of the system.

Phase space provides an effective method for science to master the system law without specific calculation.

Let the spring vibrator vibrate first, and then let the stroboscope flash against the spring vibrator every 0.05s. The vibrator will be illuminated at the moment of flashing, so as to obtain the position of the ball when flashing. The time interval between the two adjacent positions is 0.05 s. When shooting, the negative moves at a uniform speed from left to right, so a series of images of small balls and springs are left on the negative.

Or, put a spring vibrator (with a stylus) on the desktop, put a long wide paper tape under it, and then roll up the paper tape at a uniform speed on one side while the spring vibrator vibrates, so as to get an image similar to a stroboscopic photo.

The acceleration of simple harmonic motion is also inversely proportional to the displacement. For example, when riding a car, if people bump up and down with the car as simple harmonic motion, then when the car is at the lowest point, the downward displacement of people is the largest, and the acceleration is the largest. At this time, people are in a state of overweight, and the support of seats to people is the largest.

Taking the horizontal spring vibrator as an example, the kinetic energy is possessed by the ball and the potential energy is possessed by the spring. At the equilibrium position, the spring is at its original length and the potential energy is the smallest. At this time, the ball is the fastest and the kinetic energy is the largest; At the amplitudes on both sides, the shape variable of the spring is the largest, at this time, the potential energy is the largest, the velocity of the ball is zero, and the kinetic energy is zero[5].

3. First order linear differential equation

First order equations:

There are two different kinds situations to solve first order equations: separate method and integrating factor.

3.1. A. Separate method

$$\text{Let } \frac{dy}{dx} = xy \quad (1)$$

In this case, move x terms to one side, and move y terms to the other side of equation; Then, do integration on both sides of equation [6].

$$\text{So } \int \frac{dy}{y} = \int x \quad (2)$$

Doing integration:

$$\ln y = \frac{1}{2}x^2 + C \quad (3)$$

Then, keep y on the left side , and find an expression of y(x)

$$y = e^{\frac{1}{2}x^2 + C} \quad (4)$$

3.2. B. Integrating factor:

$P(x)$ and $Q(x)$, I(x) is different non-constant terms [7].

$$\frac{dy}{dx} + P(x) = Q(x) \quad (5)$$

$$\text{so, set } I(x) = e^{\int P(x)dx} \quad (6)$$

One disorder example:

$$\cos x \times \frac{dy}{dx} = y \sin x + \sin 2x \quad (7)$$

Then change it to the first form:

$$\text{So } \frac{dy}{dx} = \frac{y \sin x + \sin 2x}{\cos x} \quad (8)$$

Take it apart and turn it into two fractions

$$\frac{dy}{dx} = \frac{y \sin x}{\cos x} + \frac{\sin 2x}{\cos x} \quad (9)$$

The second fraction can be reduced to:

$$\frac{dy}{dx} = y \tan x + 2 \sin x \quad (10)$$

Move the formula on the right of the equal sign to the left.

$$\frac{dy}{dx} - y \tan x - 2 \sin x = 0 \quad (11)$$

Then

$$\frac{dy}{dx} - y \tan x = 2 \sin x \quad (12)$$

So:

$$P(x) = -\tan x \quad (13)$$

$$Q(x) = 2 \sin x \quad (14)$$

$$I(x) = e^{\int -\tan x dx} = e^{\ln \cos x} = \cos x \quad (15)$$

Finally, multiply (I(x)) on both sides of the function:

$$\frac{dy}{dx} - y \tan x = 2 \sin x \quad (16)$$

Set it into the function:

$$\frac{dy}{dx} \times \cos x - y \tan x \cos x = 2 \sin x \cos x \quad (17)$$

Then simultaneous integrate them on both sides of the equation :

This part:

$$\frac{dy}{dx} \times \cos x - y \tan x \cos x \quad (18)$$

is the result of product rule, calculate it reversely:

$$\int \frac{dy}{dx} \times \cos x - y \tan x \cos x = y \times \cos x \quad (19)$$

$$\text{So } y = \frac{\int 2 \sin x \cos x dx}{I(x)} = \frac{-\frac{1}{2} \cos 2x}{\cos x} + \frac{C}{\cos x} \text{ (C is constant term)} \quad (20)$$

4. Second differentiation equations with constant

Coefficient:

Second order:

Basic forms: $y'' + ay' + by = 0$ a and b are any constant coefficients[8].

The first step is to know the source of general solutions.

Therefore, prove a simple proposition.

To prove the general solution, assume two arbitrary solutions of the equation, and then deduce the expression of the general solution.

In this way, the general solution formula can be used to solve any specific numerical value.(5)

If two solutions of second order function are

$$y = h_1(x), y = h_2(x) \quad (21)$$

$$ph_1''(x) + pah_1'(x) + pbh_1(x) = 0 \quad (22)$$

$$qh_2''(x) + qah_2'(x) + qbh_2(x) = 0 \quad (23)$$

A simple system of linear equations of two variables

The first formula plus the second formula.

$$1 + 2 = [ph_1(x) + qh_2(x)]'' + a(ph_1(x) + qh_2(x))' + b[ph_1(x) + qh_2(x)] = 0 \quad (24)$$

So $ph_1(x) + qh_2(x)$, $p, q \in R$ is the general solution.

Solving method:

To simplify the function, presume that:

$$y = e^{\lambda x} \quad (25)$$

Solve the first, second derivative of $y = e^{\lambda x}$, and put it into basic form:

$$\text{So } \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

Extract common factor

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \quad (26)$$

solve $(\lambda^2 + a\lambda + b)$:

Because of the form of the formula, getting the solution of the quadratic equation with one variable whose value is equal to zero is needed.

In this case, the use of discriminant is the best method.

There are three situations:

Discriminant (Δ) > 0

2 different real roots λ_1, λ_2

$$\text{Set } y_1 = e^{\lambda_1 x} \quad y_2 = e^{\lambda_2 x} \quad (27)$$

So general solution is:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \quad (28)$$

Discriminant (Δ) $= 0$

2 same real roots, $y_1 = e^{\lambda x}$ is the first solution.

Assume that $y_2 = x e^{\lambda x}$ is one solution of this function.

$$\text{So } y_2 = x e^{\lambda x} \quad (29)$$

$$y_2' = e^{\lambda x} + \lambda x e^{\lambda x} \quad (30)$$

$$y_2'' = \lambda e^{\lambda x} + \lambda e^{\lambda x} + \lambda^2 x e^{\lambda x} \quad (31)$$

$$\text{Prove } y'' + ay' + by = 0 \quad (32)$$

$$\text{So } \lambda e^{\lambda x} + \lambda e^{\lambda x} + \lambda^2 x e^{\lambda x} + a(e^{\lambda x} + \lambda x e^{\lambda x}) + b(x e^{\lambda x}) \quad (33)$$

$$= (2\lambda + a) e^{\lambda x} \quad (34)$$

Because 2 same real roots, x-axis is tangent to the stationary point of the graph $(\lambda^2 + a\lambda + b) x$ value of stationary point of quadratic equation is $-\frac{b}{2a}$. It means that when $y = 0$:

$$2\lambda + a = 0 \quad (35)$$

$$\text{So } y'' + ay' + by = (2\lambda + a) e^{\lambda x} = 0 \quad (36)$$

$y_2 = x e^{\lambda x}$ is one solution of this function.

$y_1 = e^{\lambda x}$ is one solution of this function.

General solution is:

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x} \quad (37)$$

Discriminant(Δ) < 0

Imaginary roots

The application of the knowledge in the field of imaginary numbers is:

$$\text{So } \lambda_1 = \alpha + \beta i \quad \lambda_2 = \alpha - \beta i \quad (38)$$

The proof of Euler formula with Taylor series expansion is:

If function $f(x)$ has any order derivative in a field $U(x_0, r)$ of x_0 , the Taylor series of $f(x)$ at x_0 is:

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \dots + \frac{f^n(x_0)}{n!} (x - x_0)^n + \dots \quad (39)$$

Maclaurin series expansion:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \quad (40)$$

Replace x with ix:

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \dots + \frac{(ix)^n}{n!} + \dots \quad (41)$$

$$= 1 + ix - \frac{x^2}{2!} - i \frac{x^3}{3!} + \frac{x^4}{4!} + i \frac{x^5}{5!} - \frac{x^6}{6!} - i \frac{x^7}{7!} \dots \quad (42)$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + i \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots\right) \quad (43)$$

$$= \cos x + i \sin x \quad (44)$$

Euler's formula is a formula in the field of complex analysis. It associates trigonometric functions with complex exponential functions. It is named after its originator Leonhard Euler[9].

Using Taylor formula can prove that:

$$e^{ki} = \cos k + i \sin k \quad (45)$$

$$\text{Let } y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x} \quad (46)$$

$$\text{So } y_1 = e^{\lambda_1 x} = e^{(\alpha+\beta i)x} = e^{\alpha x} \times e^{\beta x i} = e^{\alpha x} \cos \beta x + i \sin \beta x \quad (47)$$

$$y_2 = e^{\lambda_2 x} = e^{(\alpha-\beta i)x} = e^{\alpha x} \times e^{-\beta x i} = e^{\alpha x} \cos \beta x - i \sin \beta x \quad (48)$$

Rationalize the results:

$$y_3 = \frac{y_1+y_2}{2} = e^{\alpha x} \cos \beta x \quad (49)$$

$$y_4 = \frac{y_1-y_2}{2i} = e^{\alpha x} \sin \beta x \quad (50)$$

So the general solution is:

$$y = C_1 e^{\alpha x} \sin \beta x + C_2 e^{\alpha x} \cos \beta x \quad (51)$$

In mathematics, Taylor formula is a formula that uses the information of a function at a certain point to describe the value taken near it. This formula comes from Taylor's theorem of calculus. Taylor's theorem describes a differentiable function. This polynomial is called Taylor's polynomial.

Taylor formula also gives the remainder.

General solution of homogeneous second differentiation equations with constant Coefficients is:

$$\text{Discriminant}(\Delta) > 0: y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

$$\text{Discriminant}(\Delta) = 0: y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

$$\text{Discriminant}(\Delta) < 0: y = C_1 e^{\alpha x} \sin \beta x + C_2 e^{\alpha x} \cos \beta x$$

5. Application

Newton's second law is also known as the "law of acceleration".

Newton's second law is known as the soul of classical mechanics. In classical mechanics, it can dominate the ever-changing movement of objects and wonderful and orderly physical phenomena. Newton's second law has a wide range of applications. It can be used to design the Taipei 101 Skyscraper that stands steadily in the clouds, and it can also be used to calculate the motion orbit of launching a rocket from the earth to the moon.

Newton's second law is a theory related to the motion of objects.

According to the core concept of the SHM:

Simple harmonic motion, or simple harmonic vibration, resonance, SHM

Basic expression of Simple harmonic motion equation:

$$X = A \cos(\omega t + \varphi) \quad (52)$$

A: amplitude of vibration

X: displacement

T: time

ω : angular frequency

φ : phase

So, given that

$$\omega = \sqrt{\frac{k}{m}} \quad (53)$$

Both sides of the equation are squared at the same time.

$$\omega^2 = \frac{k}{m} \quad (54)$$

In the simple harmonic vibration, the resultant force is equal to the elastic force under the premise of neglecting the friction force.

(Roche, John, What is mass?, Eur. J. Phys, 2005, 26)

According to the Newton's Second Law of Motion-Force and Acceleration

$$\sum F = ma = -kx \quad (55)$$

Because Acceleration(t) = the second derivative function of displacement(t)

$$mx'' = -kx \quad (56)$$

Move the formula on the right to the left.

$$mx'' + kx = 0 \quad (57)$$

Divide both sides of the equation by m at the same time

$$x'' + \frac{k}{m}x = 0 \quad (58)$$

because: $\omega^2 = \frac{k}{m}$

$$x'' + \omega^2 x = 0 \quad (59)$$

This formula can be analogized as:

$$y'' + ay' + by = 0 \quad (60)$$

Let $y = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0 \quad (61)$$

Extract the common factor.

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \quad (62)$$

$$a = 0, b = \omega^2$$

Bring the angular velocity into the equation[10].

$$\omega^2 + \lambda^2 = 0 \quad (63)$$

$$\lambda^2 = -\omega^2 \quad (64)$$

So $\lambda^2 = \sqrt{-\omega^2} = \sqrt{-1} \times \omega = \pm \omega i$

Using the previously derived formula

$$x = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x \quad (65)$$

$$= C_1 \cos \omega t + C_2 \sin \omega t \quad (66)$$

In the simple harmonic motion, when $t=0$, it satisfies that: X(displacement)=A(Amplitude), V(velocity)=0

$$V(t) = x'(t) = -\omega C_1 \sin \omega t + \omega C_2 \cos \omega t = 0 \quad (67)$$

$$\text{So } \omega C_2 = 0 \quad (68)$$

$$C_2 = 0 \quad (69)$$

$$\text{So } X = C_1 \cos \omega t \quad (70)$$

When $t=0$, $X = C_1$,

So $C_1 = A$

Finally,

$$X = A \cos \omega t \quad (71)$$

Plus the initial phase

$$X = A \cos(\omega t + \varphi) \quad (72)$$

Then, Simultaneous differentiation on both sides of the equation;

$$v(t) = \frac{dx}{dt} = -\omega A \sin \omega t + \varphi \quad (73)$$

Second derivative is:

$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 \cos \omega t + \varphi \quad (74)$$

6. Conclusion

This equation is the simplest and most common mathematical model of vibration. All values in the equation are vectors.

On the other hand, when discussing simple harmonic motion, people always solidify their thinking in one-dimensional space, and try their best to turn the general solution of simple harmonic motion into a real number solution. In fact, the general solution describes the simple harmonic motion in any dimension. The key to understanding this is to understand the two constant terms in the general solution.

These two constant terms can be said to be numbers in any number system. For example, when they are quaternions, the solution describes such a three-dimensional simple harmonic motion problem: imagine that there is a particle in a three-dimensional coordinate system, which is always affected by a force pointing to the origin. The size of the force is proportional to the distance from the particle to the origin. Given the initial position and speed of the particle, solve its equation of motion.

The problem is to change the soup without changing the medicine. The solution is the original solution, but at this time, the so-called ternary number is used to represent the initial condition, which is substituted into the general solution. In the process of solving the coefficient, strange number occurs, which is actually a quaternion.

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