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How to Find the Maximum Difference Between the Intervals of the Two Closest Intersection Points of Two Functions

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Abstract: A new method is proposed in this paper to solve the maximum difference between two functions in the two nearest intersecting intervals by using the mean value theorem (MVT). At the same time, this paper describes the theoretical basis and implementation of the method and uses examples to verify its effectiveness. The goal is to provide a simplified and efficient tool for solving complex functional analyses.

Keywords: MVT; New Mathematical Techniques; Absolute Maximum.

1. Introduction

The research idea of this paper comes from a Calculus BC exam, which focuses on the intersection of two functions [1]. For $y_1 = \sqrt{x}$ and $y_2 = \frac{2}{3}x$, we aim to find the x value that maximizes the closed area between the intersection of these two functions. It is noteworthy that the second derivative of a function does not have a symbolic change in this exam, so the students' problem-solving process is relatively simple, and the knowledge and skills required to solve the problem are not complex. They can be understood by all the students. However, we find a universally applicable mathematical problem by studying these problems: finding their maximum difference between the closest intersections of two functions. This issue is of great significance in both real-life and academic research.

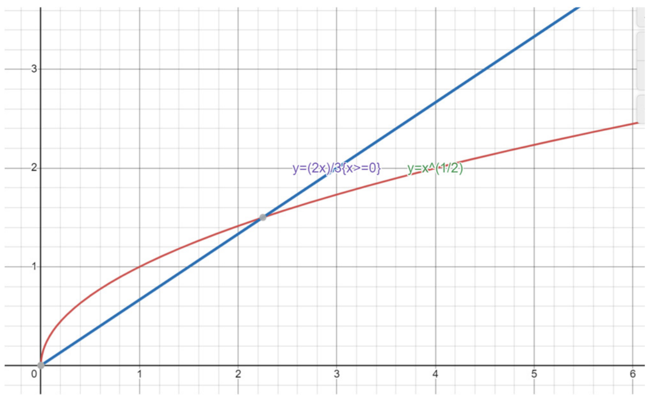

Figure 1. Mathematical graph of function

In modern mathematics education, Calculus is a basic course that covers functions, limits, derivatives, and integrals. Students need to master these concepts to understand and solve practical problems. The Calculus exam often involves the intersection of two functions, which requires students to find the intersection and analyze the relationship between the functions at the intersection. Traditional methods usually require complex calculations and graphical analysis, which is challenging for most students. Therefore, developing a simple and efficient method to solve this problem is particularly essential.

2. Background of Mathematics Education

Calculus is an essential part of higher mathematics and is widely used in science, engineering, and economics. The goal of education is to enable students to master basic concepts and computational skills as well as promote the ability to solve practical problems. However, in traditional Calculus teaching, students often have difficulties in dealing with the intersection and difference of two functions. The difficulty mainly stems from the following reasons:

(1). The complex computation: We solve the equation to find the intersection of two functions. Moreover, these equations may be nonlinear, and the solution process is cumbersome in many cases.

(2). Graphical analysis: Analyzing the difference between two functions usually requires drawing a diagram of the function and analyzing the graph, which demands students' spatial imagination and analytical abilities.

(3). Multi-step calculation: Traditional methods involve multiple steps, including derivatives, solving intersection points, comparing function values, etc., and each step can introduce calculation errors.

This paper presents a new method based on the mean value theorem to help students better understand and master the solution to such problems. It simplifies the calculations and reduces the need for graphic analysis, providing students with a more intuitive and effective way to solve problems.

2.1. Challenges of Calculus Exams

In calculus exams, students often solve problems about the maximum difference between the intersection points of two functions. This type of question not only tests students' understanding of fundamental concepts but also their ability to apply those concepts comprehensively. Specifically, these problems usually include the following aspects:

(1) Intersection solution: Students need to find the intersection of two functions, which requires them to have the ability to solve equations.

(2) Function analysis: After finding the intersection, it is necessary to analyze the relationship between two functions and the intersection to determine their maximum difference.

(3) Graphic interpretation: Students can understand the

relationship between two functions more intuitively by drawing function images. At the same time, it also increases the complexity of the problem.

For these reasons, students often feel great pressure when dealing with such problems. The method proposed in this paper aims to simplify this process and enable students to solve problems efficiently.

2.2. Practical Significance of the Study

Besides the application in the field of education, the method proposed in this paper is also of great significance in practical application. In engineering design, data analysis and economic research, it is common to analyze the maximum difference between two functions. For example:

(1) Engineering design. In mechanical design, the difference in displacement between two components can affect the operating effect of the whole system. Accurately calculating the maximum difference between these two components is very important to ensure the normal operation of the device.

(2) Data analysis. In data science, comparing the biggest differences between two data sets is an important basis for helping decision makers make more scientific decisions.

(3) Economic research. In economics, analyzing changes and differences in various economic indicators can help inform policy making.

By studying and applying the method proposed in this paper, researchers can effectively solve practical problems and improve the efficiency and accuracy of analysis. In mathematics and engineering, it is common to analyze the difference between two functions. Traditional methods are complex and laborious, especially when calculating the maximum difference between the intersection points of two functions. The aim of this research is to present a new method that simplifies this process using the Mean Value Theorem. It is suitable for academic research as well as real technical and scientific problems.

2.3. Research Background

At home and abroad, the research on the intersection and difference of two functions focuses on numerical analysis, computational mathematics, and application fields. In numerical analysis, foreign scholars have done much work on solving the intersection of functions numerically. Burden and Faires introduced the numerical method to solve the intersection of functions in detail in their classic textbook *Numerical Analysis*. In addition, they discussed the convergence and stability of these methods [2]. Press et al. put forward a solution based on polynomial interpolation and numerical integration in *Numerical Recipes: the art of scientific computing*. Experiments verify its efficiency and accuracy in dealing with complex functions [3].

In applied optimization theory, Nocedal and Wright systematically discussed nonlinear optimization methods in their book *Numerical Optimization*. They proposed a hybrid algorithm based on the interior point method and constrained optimization to solve the optimal solution of nonlinear functions [4]. This algorithm has high application value in solving the function difference problem. Goldberg introduced an optimization method based on genetic algorithm in *Genetic Algorithms in Search, Optimization, and Machine Learning*, and its effectiveness and robustness in solving complex optimization problems were verified by experiments [5]. In the practical application, Smith et al. introduced a comparison method of curative effect based on function difference in *Data Analysis in Medical Research*, verifying its accuracy and reliability in evaluating different treatment methods through experiments [6]. Jones et al. put forward a method of pollutant concentration analysis based on function difference in their paper *Application of Function Difference in Environmental Pollutant Concentration Analysis*. Experiments verify its application value in environmental monitoring [7].

3. Research Methodology

3.1. Theoretical Basis

It is vital to understand and master the relevant theoretical basis when solving the maximum difference between the two functions in the two closest intersection intervals. The methods used in this research are based on the mean value theorem and gradient analysis in Calculus. We will comprehensively introduce the theoretical basis in the following sections.

3.1.1. Mean Value Theorem (MVT)

The mean value theorem is essential in Calculus, which has a broad range of applications in analyzing the function's behavior and change rate. The basic idea of the mean value theorem is that if a function is continuous on a closed interval and differentiable on an open interval, then the average rate of change of the function in the interval is equal to its instantaneous rate of change at a certain point.

Here are the formulas and conditions of the mean value theorem. The mean value theorem can be expressed as:

Assume that function f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there is at least one point c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

In the formula, $f(a)$ and $f(b)$ are the function values of function f at points a and b; $f'(c)$ is the derivative of the function at point c.

Geometric interpretation of mean value theorem: Geometrically, the mean value theorem states that there exists a point c such that the slope of the tangent line to the function f at that point is equal to the slope of the secant line connecting the points (a, $f(a)$) and (b, $f(b)$).

Proof of the mean value theorem: The proof of the mean value theorem is usually based on Rolle's theorem. Rolle's theorem is a special case of mean value theorem.

Rolle's Theorem: Let the function be continuous on the closed interval [a, b], differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists a point c on (a, b) such that $f'(c) = 0$:

The proof steps of the mean value theorem are as follows:

(1). Define a new function $g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right)$ that represents $f(x)$ minus the secant equation.

(2). Since $g(a) = g(b) = 0$, according to Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$.

(3). Calculate $g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$. And

$g'(c)=0$ means $f'(c)=\frac{f(b)-f(a)}{b-a}$.

3.1.2. Gradient Analysis

Gradient analysis plays a key role in function comparison and optimization. When studying the maximum difference between two functions $f(x)$ and $g(x)$, the extreme points of the function within a certain interval can be determined by analyzing their derivatives (gradients).

The meaning of the derivative: The derivative $f'(x)$ represents the instantaneous rate of change or slope of the function f at a point x. For two functions $f(x)$ and $g(x)$, comparing their derivatives can reveal their relative rate of change:

(1) Derivatives are equal: If at a point x, $f'(x)=g'(x)$, then the rates of change of the two functions at that point are the same.

(2) Comparison of derivatives: If $f'(x)>g'(x)$, it means that at this point $f(x)$ grows faster than $g(x)$; if $f'(x)<g'(x)$, it means that $f(x)$ grows slower than $g(x)$.

3.1.3. Determination of Extreme Points

By solving the equations for derivatives equal to zero, $f'(x)=0$ and $g'(x)=0$, we can find the extreme points of the function. At these points, the function may reach a local maximum or minimum. By comparing the values of the two functions at these extreme points, the maximum difference between them can be found.

3.2. Specific Steps

The method in this paper combines the mean value theorem and gradient analysis to systematically find the maximum difference between two functions in the two nearest intersection intervals. The specific steps are as follows:

Step 1: Determine the intersection of the two functions:

First, we need to determine all the points where the functions $f(x)$ and $g(x)$ intersect. An intersection point is a point where the values of the two functions are equal, i.e. $f(x)=g(x)$.

Here are the specific steps:

(1). Solving equations: For simple functions, the equation $f(x)=g(x)$ can be solved by analytical method. For complex functions, it may be necessary to use numerical methods, such as Newton iterative method or dichotomy, to solve the intersection point.

(2). Verify the intersection point: We confirm that the solved point is the actual intersection point of the two functions, not the false solution.

Example: Suppose there are two functions $f(x)=x^2$ and $g(x)=\sin(x)$, and we need to solve the equation: $x^2=\sin(x)$, let x_1 and x_2 be the two intersection points.

Step 2: Calculate the secant and tangent

After finding the two nearest intersection points x_1 and x_2, we calculate the secant between the two points and find the tangent parallel to the secant. Here are the specific steps:

(1). Compute the slope of the secant line: $m=\frac{f(x_2)-f(x_1)}{x_2-x_1}$. The equation of the secant line is L: $y=m(x-x_1)+f(x_1)$

(2). Applying the mean value theorem: According to the mean value theorem, there exists a point c ($c\in(x_1,x_2)$), such that: $f'(c)=m$. Similarly, for $g(x)$, $g'(d)=m$, and we can find the points c and d such that the derivative is equal to the slope of the secant line.

Example: Using the above example, assuming x_1 and x_2 are 0.5 and 0.15, the slope of the secant line is:

$$m=\frac{(1.5)^2-(0.5)^2}{1.5-0.5}=\frac{2.25-0.25}{1}=2,$$

The equation of the secant line is L: $y=2(x-0.5)+0.25$

Step 3: Determine the relative extreme points

After finding the points under the mean value theorem, it is necessary to determine the relative extreme points of the functions $f(x)$ and $g(x)$ in the intersection interval, which may be the candidate points of the maximum difference.

Here are the specific steps:

(1). Solve the point where the derivative is zero: Derivate $f(x)$ and $g(x)$ respectively, and solve the equations $f'(x)=0$ and $g'(x)=0$ to find the extreme point.

(2). Calculate the difference in function values: Calculate the values of the two functions at these extremes and intersections and record the difference.

Example: After taking the derivative with respect to $f(x)=x^2$, the result is $f'(x)=2x$; after taking the derivative with respect to $g(x)=\sin(x)$, the result is $g'(x)=\cos(x)$. Then, solving the equation $f'(x)=2x=0$, we get the extreme point $x=0$; similarly, $g'(x)=\cos(x)=0$, we get the extreme point $x=\frac{\pi}{2}$.

Step 4: Compare the differences and find the maximum difference

Calculate the difference between $f(x)$ and $g(x)$ at all extreme points and intersections. Compare these differences to find the maximum.

Here are the specific steps:

(1). Calculate the difference: Calculate the difference at each extreme point and intersection point x_i: $\Delta(x_i)=|f(x_i)-g(x_i)|$

(2). Find the maximum difference: Compare all the differences to determine the maximum difference and the corresponding points.

For example, compute the difference between $f(x)=x^2$ and $g(x)=\sin(x)$ at $x_1=0.5$ and $x_2=1.5$, and the extreme points $x=0$ and $x=\frac{\pi}{2}$:

$x_1=0.5$: $f(0.5)=0.25, g(0.5)=\sin(0.5)\approx 0.479$

difference $\Delta(0.5)\approx|0.25-0.479|=0.229$。

$$x_2 = 1.5 : f(1.5) = 2.25, g(1.5) = \sin(1.5) \approx 0.997$$

difference $\Delta(1.5) \approx |2.25 - 0.997| = 1.253$.

$$x = 0 : f(0) = 0, g(0) = \sin(0) = 0,$$

difference $\Delta(0) = 0$.

$$x = \frac{\pi}{2} : f(\frac{\pi}{2}) \approx 2.47, g(\frac{\pi}{2}) = 1,$$

difference $\Delta(\frac{\pi}{2}) \approx |2.47 - 1| = 1.47$

Compare the above differences and get the maximum difference: $\Delta(\frac{\pi}{2}) \approx 1.47$.

4. Practical Application

The method proposed in this paper to solve the maximum difference between two functions in their two closest intersection intervals is of great theoretical importance and is widely used in practice. The following are several typical application cases:

Different asset prices (such as stocks, bonds, futures, etc.) often show different fluctuation patterns in the financial market. By analyzing the price fluctuation of these assets, we can help investors make more reasonable investment strategies. The method proposed in this paper allows us to compare the maximum difference between two asset prices within a certain period, thus identifying extreme market fluctuations.

For the engineering field, accuracy and error analysis are very important. The accuracy and stability of a measurement method can be assessed by comparing the maximum difference in a specific interval for data obtained from two different measurement methods. The method proposed in this paper can be used to determine the maximum error of two measurement results in the intersection interval, thus evaluating the reliability of the measurement method.

For medical research, it is a common demand to compare the efficacy of treatment methods. By analyzing the effect curves of different treatment methods, we can find the maximum difference between the two treatment methods in a specific period and help doctors choose the most effective treatment scheme. When comparing the maximum difference between the two curative effect curves, we can use the method in this paper to evaluate the difference in therapeutic effect.

For environmental science research, changes in the concentration of pollutants can reflect changes in the quality of the environment. By comparing the maximum concentration differences of different pollutants over some time, the main causes of environmental pollution can be identified. When comparing the maximum difference between two pollutant concentration curves, the method in this paper can be used to evaluate the impact of pollutants on the environment.

5. Conclusion

A new method based on the mean value theorem is proposed in this paper to solve the maximum difference between two functions in the interval closest to the intersection point. Through detailed theoretical analysis and practical example checks, we prove the effectiveness and practicality of our method. Future research can further optimize our method and explore its application to more practical problems.

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