

A New Quantitative Characterization of the Monster Group and the Baby Monster Group

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Abstract: Let G be a finite group, $\pi(G)$ the set of prime divisors of the order of G , p_m the greatest element in $\pi(G)$, and $\pi_{p_m}(G)$ the set of orders of centralizers of elements of p_m maximal order in G . This paper presents a novel and comprehensive quantitative characterization of the Monster group F_1 and the Baby Monster group F_2 , achieved through the employment of the even-order component inherent to a group, in conjunction with the aggregate set comprising the orders of centralizers associated with its elements of maximal order. We establish that a finite group G , under the specified conditions, is necessarily isomorphic to the sporadic simple group M under consideration, thereby furnishing a rigorous identification within the classification framework of finite simple groups. if and only if the following two conditions hold: (1) the even-order component of m_1 is identical to that of G ; (2) $\pi_{p_m}(G) = \pi_{p_m}(M)$, where M is either the Monster group F_1 or the Baby Monster group F_2 .

Keywords: Finite Simple Group; Monster Group; Order Component; Centralizer.

1. Introduction

Since the inception of research in group theory, numerous scholars have been fascinated by the characterization problem of finite simple groups. Finite simple groups hold a position in group theory analogous to that of prime numbers among integers. In the 1980s, the Classification Theorem of Finite Simple Groups was conclusively proved. Following the complete proof of the "Shi Wujie Conjecture" proposed by Professor Shi Wujie, an increasing number of researchers have sought to characterize finite simple groups under fewer conditions. Numerous examples exist in this regard, such as Lagrange's Theorem proposed in the 18th century, the Sylow theorems introduced by the Norwegian mathematician Peter Ludwig Mejdell Sylow in 1872, and Burnside's lemma proposed by William Burnside in 1897. This often involves employing diverse knowledge and tools, such as group representation theory, module theory, and sophisticated combinatorial methods. Through quantitative characterizations, we can gain a deeper understanding of the structure of finite simple groups, unveil their inherent patterns and characteristics, and provide a foundation for further research and applications. Group theorists are dedicated to seeking a unified method to characterize all series of finite simple groups holistically. In 1987, Fields Medalist Professor J. G. Thompson proposed the famous Thompson Conjecture: Given that G and H are groups sharing an identical order type, if G is solvable, does it necessarily follow that H is likewise solvable?

In 2012, Professor Li Guan He, along with his collaborators, established a characterization of finite simple groups by leveraging the properties associated with the orders of elements of maximal order, the total number of such elements, and the structure of their centralizers. Their work primarily

characterized simple K_3 - groups, sporadic simple groups, some alternating simple groups, some simple K_4 - groups of Lie type, and certain other families. In 2016, Professor Yanwei Gao obtained a new characterization of sporadic simple groups using the order of elements of maximal order: Let G be a finite group, and let $K_1(G)$ denote the order of an element of maximal order in G . It was proved that every sporadic simple group G can be uniquely characterized by $|G|$ and $K_1(G)$. In 2019, Hu Peijie investigated the influence of group order and the orders of higher-order elements on group structure, primarily using these two factors to characterize the automorphism groups of simple K_4 - groups, achieving notable results. In 2012, Professor Yanxiong Yan completed the characterization of $O_{10}^+(2)$ using OD - . In 2015, Professor Yanxiong Yan proceeded to characterize some alternating and symmetric simple groups. In 2016, Zhang Mengmeng, by utilizing the group order and the degree sequence of the prime graph, provided a unified proof that groups possessing a disconnected prime graph are characterizable. Furthermore, he established that such groups are also 2-characterizable and admit a two-step characterization. By employing the group order and the degree pattern of the prime graph, he demonstrated that the groups, are characterizable, and likewise concluded that, are also characterizable and amenable to a two-step characterization. It was further deduced that all finite non-abelian simple groups whose prime divisors do not exceed 17 are characterizable as well. The proofs of these results provide additional validation for the correctness of the WJSI conjecture.

Let G be an arbitrary finite group. The prime graph

associated with G , commonly denoted by $\Gamma(G)$ and originally conceived by K. W. Gruenberg and O. Kegel, is defined through the following construction: The vertex set of this graph is taken to be the set of all prime numbers that divide the order of G , which is typically denoted by $\pi(G)$. Subsequently, for any two distinct vertices p and q belonging to $\pi(G)$, an edge is drawn between them—indicating adjacency—if and only if the group G contains at least one element whose order is exactly the product pq . In other words, the existence of an element whose order equals the least common multiple of p and q (which, owing to the distinctness and primality of the two numbers, reduces simply to their product) is both necessary and sufficient for the two vertices to be adjacent in $\Gamma(G)$. [10].

In 1996, Professor Guiyun Chen first introduced the definition of order components of a group in literature [2]: Let G be a finite group, and let $\ell(G)$ denote the number of connected components of the prime graph $\Gamma(G)$. Subsequently, many scholars began to study finite simple groups using order components $OC(G)$ [4-12]. For example, prominent group theorists including Professor Guiyun Chen and Professor Huaguo Shi have undertaken a rigorous and systematic investigation into a particular problem, the precise formulation of which is delineated as follows:

In the realm of finite group theory, consider an arbitrary finite group G alongside a non-abelian simple group S . Under the condition that the invariant denoted by $OC(G)$ —whatever its precise interpretation may be—is exactly equal to the corresponding invariant $OC(S)$, a natural and profound question arises: does this equality necessarily compel the conclusion that G is isomorphic to S ?

In 1962, the distinguished group theorist Richard Brauer pioneered the innovative methodology of employing centralizers of involutions as distinctive structural markers to characterize special elements within finite simple groups. This groundbreaking approach leveraged the fact that involutions—elements of order two—often possess centralizers with particularly restrictive and informative algebraic structures, thereby serving as powerful tools for distinguishing among various classes of simple groups. Subsequently, a growing number of scholars extended this line of inquiry by directing their attention toward the centralizers of elements whose orders are odd, recognizing that such centralizers also encode substantial information about the underlying group architecture and can be instrumental in achieving more comprehensive classification results.

Consequently, the present work sets forth the following principal investigative aim: to furnish a characterization of finite simple groups by means of an analysis that incorporates both the even-order component of a group and the ensemble of orders pertaining to the centralizers of its elements possessing maximal order. The main result can be summarized as follows: Given a finite group G and a sporadic simple group S taken from the set $\{F_1, F_2\}$, the isomorphism $G \cong S$ holds if and only if the subsequent two criteria are simultaneously fulfilled:

- (1) $m_1(G) = m_1(M)$;
- (2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

The Monster Group F_1 is the largest sporadic simple group, proposed by German mathematicians Fischer and Griess in 1973. Its number of elements is approximately 8×10^{53} , requiring at least a 196,883-dimensional space for representation, and it exists as the automorphism group of the Griess algebra. After discovering the F_{24} series of groups in 1971, Fischer conjectured the existence of the Monster group, which was realized through Griess's construction of the 196,884-dimensional Griess algebra. In subsequent research, the team of Frenkel reconstructed the symmetry of the Monster group via Vertex Operator Algebras, while Wilson described its group representation by generating $196,882 \times 196,882$ matrices using computers. These achievements advanced the classification of finite simple groups and became representative case studies in 20th-century group theory.

The Monster group is directly related to the "Monstrous Moonshine Conjecture," where its irreducible representations exhibit profound connections with the modular function $J - \text{invariant}$. In 1992, Borchers proved the Moonshine Conjecture using the No-ghost theorem from string theory, revealing links between group theory, modular functions, and string theory. The Baby Monster group F_2 is a subgroup of the Monster group F_1 .

2. Preliminary Lemmas

The following result delineates the structural characterization of a finite group under the condition that its prime graph is disconnected, i.e., when the number of connected components $t(G)$ satisfies $t(G) = 2$.

Lemma 2.1 [1]. Let G be a finite group whose prime graph $\Gamma(G)$ is disconnected, meaning that $t(G) = 2$. Then the group G necessarily exhibits one of the following structural configurations:

- (1) G is either a Frobenius group or a 2-Frobenius group;
- (2) Alternatively, G admits a subnormal series of the form

$$1 \leq H \leq K \leq G$$

with the property that the factor group K/H constitutes a nilpotent $\pi_1(G)$ -group, while both H and the quotient G/K are solvable groups whose prime divisors lie entirely within the complement of $\pi_1(G)$ (i.e., they are $\pi_1(G)$ -groups). Moreover, the factor K/H is, in fact, a non-abelian simple group satisfying $K/H \cong S$.

Remark 2.1. Let G be a finite group. G is called a 2-group if G has a normal series $1 \leq H \leq K \leq G$ such that G/H and K/H are $\pi_1(G)$ -groups with kernels K and H , respectively.

Lemma 2.2 [1]. Let G be a Frobenius group of even order,

admitting a Frobenius kernel K and a corresponding Frobenius complement H . In this classical group-theoretic configuration, the order of G factorizes as the product $T(G) = \{\pi(H), \pi(K)\}$, where the orders of K and H are necessarily coprime, i.e., $t(G) = 2$. The algebraic structure of the complement H is then constrained to one of the following mutually exclusive cases, depending on the parity of its order and on whether G is solvable:

(1) In the scenario where H possesses even order, i.e., $2 \in \pi(H)$, any Sylow 2-subgroup contained within H is necessarily cyclic.

(2) Conversely, if K is of odd order, i.e., $2 \in \pi(K)$, then H itself must be abelian.

Furthermore, under the additional hypothesis that G is solvable, all Sylow subgroups of K corresponding to odd primes are likewise cyclic, while the Sylow 2-subgroup of K is either cyclic or of generalized quaternion type.

In the event that G fails to be solvable, there exists a subgroup K_0 satisfying $K_0 \cong Z \times SL(2, 5)$, where Z denotes a cyclic group whose order is coprime to 30, and furthermore the Sylow 2-subgroup of Z is constrained to be cyclic.

Lemma 2.3 [1]. Let G be a finite group of even order that admits the structure of a 2-Frobenius group. Then G possesses a normal series of the form $\pi(H) \cup \pi(G/K) = \pi_1(G)$, satisfying the following conditions: the quotient groups G/K and K/H are both cyclic, and moreover G/K divides $\text{Aut}(K/H)$; additionally, both H and G/K are cyclic groups. In particular, the inequality $|G/K| < |K/H|$ holds, and the group G is necessarily solvable.

Lemma 2.4 [13]. Let G be a finite group and suppose that N is a normal subgroup of G . Then the collection of maximal centralizers associated with the quotient group G/N is always contained within the corresponding collection of maximal centralizers of G itself; i.e., the inclusion $G/(NC_G(N)) \leq \text{Out}(N)$ holds as a set-theoretic containment. In the special case where the normal subgroup N reduces to the trivial subgroup (i.e.,

$C_G(N) = 1$), one obtains the stronger conclusion that the two collections coincide, yielding the equality $N(G) = \text{Aut}(G)$.

Lemma 2.5. In accordance with the renowned Classification Theorem of Finite Simple Groups—one of the most profound achievements in twentieth-century mathematics—every finite simple group necessarily falls into one of the following mutually exclusive categories:

(i) groups of prime cyclic order, i.e., cyclic groups whose order is a prime number;

(ii) alternating groups of degree at least five, denoted A_n for 5;

(iii) simple groups belonging to the classical families,

which arise from bilinear forms over finite fields;

(iv) exceptional simple groups of Lie type, corresponding to the exceptional root systems and including the Tits group when appropriate;

(v) the twenty-six sporadic simple groups, each of which does not conform to any of the preceding infinite families and whose existence is, in a sense, an accident of the classification.

3. Main Theorems and Their Proofs

Research on the Monster group holds significant importance in the field of group theory, with many areas of study involving the structural analysis of the Monster groups. The Monster group was proposed by the German mathematician Fischer in 1973, with mathematician Griess joining the research subsequently. Initially, the focus was on the Baby Monster group. During this research, they discovered that the Baby Monster group is the centralizer of an involution within a larger group, namely the Monster group. An interesting episode involved the initial proposal of an intermediate "Midnight Monster" group between these two, which was later proven not to exist. Since then, many group theorists have dedicated efforts to studying the structures of these two groups.

Theorem 3.1

Let G denote an arbitrary finite group, and suppose that the group M under consideration is specifically the Monster sporadic simple group, denoted by F_1 . Under this hypothesis, the group G is isomorphic to M if and only if the following two necessary and sufficient criteria are simultaneously fulfilled:

$$(1) m_1(G) = m_1(M);$$

$$(2) \pi_{p_m}(G) = \pi_{p_m}(M).$$

Proof. Necessity is evident: if $G \cong M$, the two simple groups correspond one-to-one in terms of order, multiplication rules, etc., making conditions (1) and (2) trivially true. Therefore, it suffices to prove sufficiency.

Under the hypothesis that condition (1) is satisfied, it follows that the even-order component of the group G necessarily coincides with the corresponding even-order component belonging to the group M ; in other words, both are precisely the same invariant, which can be denoted compactly by the notation $OC(M)$. This equality implies that the structural parameter encoding the even-order part of G is identical to that of M , and consequently the common value is exactly the even-order component associated with the Monster sporadic simple group M , namely $OC(M)$. Under the stipulation that condition (2) is fulfilled, the collection of orders corresponding to the centralizers of elements having maximal order within G is exactly identical to the analogous collection within M ; in both cases, this set consists solely of the single prime number 71. As a direct consequence of this equality, it becomes evident that the number of connected components in the prime graph of G , denoted $t(G)$, is at least two. This observation, in turn, leads to the conclusion that the algebraic structure of G must conform to the description provided in Lemma 2.1,

thereby invoking the structural constraints delineated therein.

If G is a Frobenius group, then by Lemma 2.2, $G = HK$, where K is the kernel and H is the complement. Moreover, $t(G) = 2$, and the two connected components correspond precisely to the order components of K and H . We consider two cases:

(1) Suppose, for the sake of analysis, that the prime 2 is included among the prime divisors constituting the order component associated with the normal subgroup K . Given that K is nilpotent by hypothesis, it follows necessarily that K must itself be a 2-group, i.e., its order is a power of 2. Under this configuration, the complementary subgroup H is consequently of odd order, and furthermore, the order of H divides the order of the automorphism group of K , denoted $\text{Aut}(K)$. However, it is evident from the prevailing conditions that the prime 71 belongs to the set of prime divisors of the order component of H , which implies that 71 necessarily divides H . This requirement imposes that 71 also divides $\text{Aut}(K)$, given the divisibility relation $H \mid \text{Aut}(K)$. But $\text{Aut}(K)$ itself divides the order of the automorphism group of a finite 2-group, and such orders are well known to be powers of 2, thereby excluding the possibility of any odd prime divisor, let alone 71. This direct contradiction between the necessity that 71 divide H and the inherent structure of $\text{Aut}(K)$ forces the conclusion that the initial assumption is untenable. Hence, the prime 2 cannot belong to the set of prime divisors of the order component of K .

(2) Suppose, alternatively, that the prime 2 is contained within the set of prime divisors constituting the order component associated with the subgroup H . Under this assumption, consider a Sylow 2-subgroup P of H . By virtue of the group-theoretic configuration, P naturally acts by conjugation on a Sylow 71-subgroup of K , given the structural relationships inherent in the group. This action, if nontrivial, would give rise to the existence of an element in G whose order is precisely the product $2 \cdot 71 = 142$, as the semidirect product structure would necessarily yield an element combining the action of an involution from P with an element of order 71 from K . However, such a conclusion stands in direct contradiction to the previously established condition that the centralizer of any element of maximal order within G has order exactly 71, thereby precluding the existence of any element whose order exceeds 71 or involves prime factors other than 71 in such a manner. Consequently, this line of reasoning forces the rejection of the hypothesis that 2 belongs to the order component of H . Moreover, this contradiction demonstrates that G cannot admit the structure of a Frobenius group, as the assumed configuration would otherwise inevitably lead to the prohibited element of order 142.

Suppose, for the purpose of argument, that G admits the structure of a 2-Frobenius group. In accordance with the structural characterization provided by Lemma 2.3, such a group necessarily possesses a normal series of the form $1 \leq H \leq K \leq G$, where the prime graph of G has

exactly two connected components, i.e., $t(G) = 2$. Within this series, both quotients G/K and K/H are cyclic groups, and furthermore, the even-order component of G , denoted $OC(G)$, decomposes as the union. Given that the prime 71 does not appear among the prime divisors constituting $OC(G)$, it follows necessarily that 71 must reside within the order component associated with the normal subgroup K , i.e., $71 \in OC(K)$. Consequently, K contains an element whose order is exactly 71. Now consider the action of this element of order 71, viewed as an automorphism via conjugation, upon a Sylow 2-subgroup of H . Such an action, if nontrivial, would inevitably give rise to an element within G whose order is the product $2 \cdot 71 = 142$, as the semidirect product structure inherent in the 2-Frobenius configuration would combine the 71-cycle with an involution from the Sylow 2-subgroup. However, the existence of an element of order 142 in G directly contradicts the previously established condition that the centralizer of any element of maximal order in G has order exactly 71, thereby precluding the possibility of any element whose order involves prime factors beyond 71 in such a multiplicative fashion. This contradiction renders the initial hypothesis untenable, and we are forced to conclude that G cannot be isomorphic to a 2-Frobenius group under the given constraints.

Consequently, the structural configuration of G must necessarily conform to the second case delineated in Lemma 2.1. That is, G admits a normal series of the form $1 \leq H \leq K \leq G$, wherein the factor group K/H constitutes a non-abelian simple group. Under this framework, suppose, for the sake of contradiction, that the normal subgroup H is nontrivial. In such a scenario, one may consider an element of order 71 residing within H . By virtue of the conjugacy action of this element upon a Sylow 2-subgroup of K , the inherent group structure would inevitably produce an element whose order is the product $2 \cdot 71 = 142$, or at the very least would perturb the structure of centralizers in such a manner that the centralizer of an element of maximal order in G could no longer have order exactly 71. This would directly contravene the previously established condition that the centralizer of any maximal order element in G possesses order precisely 71. Hence, the hypothesis that H is nontrivial leads to an irreconcilable contradiction, forcing the conclusion that H must be trivial.

With H thus reduced to the identity subgroup, it follows that G contains a normal non-abelian simple subgroup, which we may denote by $H = K$. Moreover, the set of prime divisors constituting the order components of the quotient group K/H is entirely contained within the set of prime divisors of the even-order component $OC(G)$, and furthermore coincides precisely with the corresponding set associated with the simple group M . In addition, the prime 71 necessarily belongs to the set of prime divisors of the order components of H . A thorough examination of the classification results documented in the literature, specifically

reference [3], reveals that under these constraints, the possible isomorphism types for the simple group H are severely restricted. Indeed, according to the analysis presented therein, H can be isomorphic only to either the alternating group $A_1(70)$ of degree 71 or the projective special linear group F_1 . These two candidates exhaust the possibilities consistent with the structural conditions imposed upon G .

If $K \cong A_1(70)$, then $m_1(K) \mid |(G/K)| \mid |Out(A_1(70))| = 2$. Calculating G from the given conditions leads to a contradiction. Therefore, $K \cong A_1(70)$.

As a direct consequence of the preceding chain of reasoning, we arrive at the conclusion that the simple normal subgroup K must be isomorphic to the projective special linear group F_1 . Given this determination, a straightforward calculation reveals that the order of K is given. From this, it follows that the index of K in G , i.e., the order of the quotient group G/K , is necessarily equal to H/K , where H denotes the target sporadic simple group under consideration. By standard group-theoretic principles, the quotient G/K embeds naturally into the outer automorphism group of K , denoted $Out(K)$. It is well established that $Out(F_1) = 2$, so the possibilities for G/K are severely restricted. Invoking Lemma 2.4, which relates the structure of maximal centralizers to such extensions, we deduce that either G/K is trivial, i.e., G/K , or G/K is isomorphic to the cyclic group of order 2, denoted C_2 . Thus, we conclude that $G \cong F_1$.

Theorem 3.2

We now turn our attention to the case where G denotes an arbitrary finite group and M is taken to be the sporadic simple group known as the Baby Monster, conventionally denoted by the symbol F_2 . Under this configuration, the group G is isomorphic to the aforementioned sporadic simple group M precisely when the subsequent two conditions are simultaneously satisfied:

- (1) $m_1(G) = m_1(M)$;
- (2) $\pi_{p_m}(G) = \pi_{p_m}(M)$.

Proof. The necessity is obvious as in Theorem 3.1. We prove sufficiency.

Under condition (1), $OC(G) = OC(M)$. Under condition (2), $\pi_{p_m}(G) = \pi_{p_m}(M) = 47$. Thus, $t(G) = 2$, and G satisfies Lemma 2.1.

If G is Frobenius, a similar case analysis as in Theorem 3.1 leads to contradictions: If 2 is in the kernel's component, 47 dividing the complement's order conflicts with automorphism group orders of 2-groups. If 2 is in the complement's component, an element of order 94 would exist,

contradicting the maximal centralizer order being 47.

If we suppose, for the sake of argument, that G admits the structure of a 2-Frobenius group, then an application of Lemma 2.3, in conjunction with a detailed analysis of the group-theoretic configuration, leads to the following consideration: taking an element of order 47 (whose existence is guaranteed by the structure of the relevant component) and allowing it to act by conjugation upon a Sylow 2-subgroup of the appropriate intermediate subgroup would inevitably produce, through the inherent action, an element whose order is precisely the product $47 * 2 = 94$. The emergence of such an element of order 94 within G would stand in direct violation of the previously established constraints, thereby yielding yet another irreconcilable contradiction. Consequently, this line of reasoning further reinforces the conclusion that G cannot be isomorphic to a 2-Frobenius group under the given hypotheses.

Based on the foregoing chain of reasoning, and in accordance with the structural dichotomy established in Lemma 2.1, it follows that the group G must exhibit precisely the configuration delineated in part (2) of that lemma, within which the lower normal subgroup H is necessarily trivial. Its normal simple subgroup K must satisfy that the prime divisors of $OC(G/K)$ are within $OC(G)$ and equal those of $\backslash(S\backslash)$, and 47 belongs to the prime divisors of $OC(K)$. By [3], K is either $A_1(46)$ or F_2 .

$A_1(46)$ leads to order contradictions. Hence $K \cong A_1(46)$. Then $|G/K| = |H/K|$. Since $Out(A_1(46)) = 2$, by Lemma 2.4, $G \cong F_2$ or $G \cong Aut(F_2)$. G/K gives the wrong centralizer set. Therefore $G/K \cong Aut(F_2)$. The unique appropriate extension yielding the Baby Monster's order and properties forces $G \cong F_2$.

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References

- [1] Chen, G. Y., The structure of Frobenius groups and 2-Frobenius groups[J], Journal of Southwest China Normal University (Natural Science Edition), 20(5)(1995), 485-487. (in Chinese).
- [2] Chen, G. Y., A New Characterization of Sporadic Simple Groups [J], Algebra Colloquium, 3(1)(1996), 49-58.
- [3] Herzog, M., On finite simple groups of order divisible by three primes only[J], Journal of Algebra, 10(1968), 383-388.
- [4] Iranmanesh, A., Alavi, S. H., Khosravi, B., A characterization of $PSL(3, q)$ where q is an odd prime power[J], Journal of Pure and Applied Algebra, 170(2002), 243-254.
- [5] Iranmanesh, A., Alavi, S. H., Khosravi, B., A characterization of $PSL(3, q)$ for $q = 2^m$ [J], Acta Mathematica Sinica, English Series, 18 (2002), 463-472.

- [6] Iranmanesh, A., Alavi, S. H., A characterization of simple groups $\text{PSL}(5, q)$ [J], Bulletin of the Australian Mathematical Society, 65 (2002), 211–222.
- [7] Iranmanesh, A., Khosravi, B., A characterization of $F_4(q)$ where q is even[J], Far East Journal of Mathematical Sciences, 2 (2000), 853–859.
- [8] Kondrat'ev, A. S., Prime graph components of finite simple groups[J], Mathematics of the USSR-Sbornik, 67(2)(1990), 235-247.
- [9] Khosravi, B., Khosravi, B., A characterization of $E_6(q)$ [J], Algebras, Groups and Geometries, 19 (2002), 225–243.
- [10] Williams, J. S., Prime graph components of finite groups[J], Journal of Algebra, 69 (1981), 487–513.
- [11] Shi, W., A new characterization of the sporadic simple groups[J], in: Group Theory, Proceedings of the Singapore Group Theory Conference, de Gruyter, Berlin, 1989, pp. 531–540.
- [12] Shi, W., Bi, J., A new characterization of the alternating groups[J], Southeast Asian Bulletin of Mathematics, 16 (1992), 81–90.
- [13] Wang, L. H., A new characterization of simple K_3 groups[J], Northeastern Mathematical Journal, 17(2)(2001), 205-209. (in Chinese).