

A Note on S -Prüfer Domains

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Abstract: S -Prüfer domains are introduced and described in the literature by a multiplicatively closed set S . A domain R is said to be S -Prüfer if every f.g. S -ideal of R is projective, where an ideal I of R is called an S -ideal if I contains an element of S . This paper proves that if R is an S -Prüfer domain, then R/I is a coherent ring for any finitely generated S -ideal I of R ; if every f.g. S -ideal I of a domain R is f.p., then R/I is also f.g.. On this basis, two equivalent characterizations of S -Prüfer domains are given, i.e., R is an S -Prüfer domain if and only if R/I is an IF ring for any f.g. S -ideal I of R ; if and only if R is integrally closed in R_S , every f.g. S -ideal I of R is f.p., and every f.g. S -ideal of R is a v -ideal. In addition, this paper presents a equivalent characterization of $\sum_{i=1}^n (a_i \otimes b_i) = 0$ ($a_i \in A, b_i \in B$) in $A \otimes_R B$, i.e., for R -modules A and B , $\sum_{i=1}^n (a_i \otimes b_i) = 0$ ($a_i \in A, b_i \in B$) if and only if there exist $c_j \in B$ ($j=1, 2, \dots, m$), a positive integer $n' \geq n$, $a_{n+1}, \dots, a_{n'} \in A$, and $r_{ij} \in R$ ($i=1, 2, \dots, n'; j=1, 2, \dots, m$) such that $b_i = \sum_{j=1}^m r_{ij} c_j$ and $0 = \sum_{j=1}^m r_{ij} c_j$ for any i ($i=1, 2, \dots, n$) and any t ($t=n+1, \dots, n'$), and $\sum_{i=1}^{n'} a_i r_{ij} = 0$ for any j .

Keywords: S -Prüfer Domain; S -ideal; IF Ring; Tensor Product.

1. Introduction

All rings mentioned in this paper are commutative rings with identity. Prüfer domains are classical rings in commutative algebra. A Prüfer domain refers to a domain in which all finitely generated (for short, f.g.) ideals are projective (equivalently invertible). The classical results on Prüfer domains are of great guiding significance for an intensive investigation of rings. For systematic and rich properties of Prüfer domains, one may refer to the monograph Prüfer Domains [2]. In order to extend the classical theory of Prüfer domains to a wider range of applications, many scholars have investigated generalized Prüfer rings under weaker conditions by weakening the ideal conditions or projectivity conditions. For example, Prüfer rings (rings where all f.g. regular ideals are invertible)[3], G -Prüfer domains (domains where all nonzero f.g. ideal is G -projective)[4], and PvMD (domains in which every nonzero f.g. ideal is w -invertible)[5].

It is a common practice to extend classical ring classes with the aid of a multiplicatively closed set S . Suppose S is a nonempty subset of a ring R . A subset S is called a multiplicatively closed set of R if the below conditions hold: (1) $1 \in S$; (2) if $s_1, s_2 \in S$, then $s_1 s_2 \in S$. For instance, Anderson et al.[6] introduced and investigated S -finite modules and S -Noetherian rings in 2002; Kim et al. [7] introduced and investigated S -SM domains in 2004; Bennis and El Hajoui[8] introduced and investigated S -finitely presented modules and S -coherent rings in 2018, etc. It is worth noting that scholars often extend classical results of domains to commutative rings by using the multiplicatively closed set which consisting of regular elements. For example, regular Noetherian rings (each regular ideal is f.g.[9]) and Dedekind rings (rings with every regular ideal invertible[10]), etc. An

ideal I of R is said to be regular if there exists some regular element s such that $s \in I$. Jamil et al.[11] followed the idea of regular ideals, and introduced S -Noetherian rings (every S -ideal of R is f.g.). It was also demonstrated that S -Noetherian rings and S -Noetherian ones are different, S -Noetherian rings and Noetherian ones are different in 2025. For an ideal I of R , if there exists some element $s \in S$ such that $s \in I$, then I is called an S -ideal of R . Recently, Rafique et al.[1] defined S -Prüfer domains, i.e., a domain in which every f.g. S -ideal is invertible. They also gave some equivalent characterizations of S -Prüfer domains in reference[1]. As an illustration, a domain R belongs to the class of S -Prüfer domains precisely when all f.g. S -modules are projective, all S -modules are flat, R is integrally closed in R_S and R/aR forms an IF ring for each nonunit element $a \in S$. Recall that an R -module M is said to be an S -module if there exist a free module F and an element $s \in S$ such that $sF \subseteq M \subseteq F$ [1]. A ring R is referred to as an IF ring when all injective R -modules are flat modules[12]. As is well known, Prüfer domains and IF rings also have a classical result, i.e., a domain R is Prüfer precisely when the quotient ring R/I is an IF ring for each f.g. ideal I of R [13, Proposition 5.3]. Then there is a natural question: does an S -Prüfer domain have the similar equivalent characterization? The answer is yes. This paper, based on reference[1], supplements two equivalent characterizations of S -Prüfer domains, i.e., R is an S -Prüfer domain if and only if R/I is an IF ring for any f.g. S -ideal I of R ; if and only if R is integrally closed in R_S and every f.g. S -ideal of R is finitely presented (for short, f.p.), and every f.g. S -ideal of R is a v -ideal.

Flat modules play an important role in the characterization of rings. The definitions of flat modules or generalized flat

modules rely on an important functor in homological algebra: tensor product \otimes . Suppose A and B be R -modules. In this paper, we present equivalent characterizations of $\sum_{i=1}^n (a_i \otimes b_i) = 0$ in $A \otimes_R B$, i.e., suppose A and B be R -modules, then $\sum_{i=1}^n (a_i \otimes b_i) = 0 (a_i \in A, b_i \in B)$ if and only if there exist $c_j \in B (j=1, 2, \dots, m)$, a positive integer $n' \geq n$, $a_{n+1}, \dots, a_{n'} \in A$, and $r_{ij} \in R (i=1, 2, \dots, n'; j=1, 2, \dots, m)$ such that $b_i = \sum_{j=1}^m r_{ij} c_j$ and $0 = \sum_{j=1}^m r_{tj} c_j$ for any $i (i=1, 2, \dots, n)$ and any $t (t=n+1, \dots, n')$, and $\sum_{i=1}^{n'} a_i r_{ij} = 0$ for any j . We give an example to show that the above equivalent characterization of $\sum_{i=1}^n (a_i \otimes b_i) = 0$ does not necessarily hold if $n' = n$.

2. Characterizations of S-Prüfer Domains

Suppose S is a multiplicatively closed subset of a ring R , and I is an ideal of R . If some element $s \in S$ is contained in I , then I is defined as an S -ideal of R . An S -Prüfer domain need not be a Prüfer domain [1, Example 2.2]. Reference [1] gave some equivalent characterizations of S -Prüfer domains. The purpose of this paper is to supplement two equivalent characterizations of S -Prüfer domains based on reference [1], i.e., Theorem 2.2 and Theorem 2.4.

Proposition 2.1 Let R be an S -Prüfer domain, then R/I is a coherent ring for any f.g. S -ideal I of R .

Proof Let I denote a f.g. S -ideal of R . For the purpose of proving that R/I is a coherent ring, it suffices to show that any f.g. ideal of R/I is f.p.. Suppose J/I be an f.g. ideal of R/I , where J is an ideal of R that contains I . So we can obtain the R -module exact sequence

$$0 \rightarrow I \rightarrow J \rightarrow J/I \rightarrow 0.$$

Since I is f.g. and J/I is a f.g. R -module, then J is f.g.. Since R is a S -Prüfer domain, then each f.g. S -ideal of R is invertible. Since invertible ideals of a domain are projective, then every f.g. S -ideal of R is projective. Therefore, J is f.g. and projective. Moreover, since f.g. projective modules are f.p., so J is f.p.. According to the fact that I is f.g. and J is f.g. projective, then J/I is f.p. as an R -module. We now consider a commutative diagram in which all rows and columns are exact:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \bigoplus_{i=1}^k Ie_i & \rightarrow & \bigoplus_{i=1}^k Ie_i & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & K & \rightarrow & \bigoplus_{i=1}^k Re_i & \xrightarrow{\varphi} & J/I \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \frac{K}{\bigoplus_{i=1}^k Ie_i} & \rightarrow & \frac{\bigoplus_{i=1}^k Re_i}{\bigoplus_{i=1}^k Ie_i} & \rightarrow & C \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0.
 \end{array}$$

Take any element $a_1 e_1 + a_2 e_2 + \dots + a_k e_k$ in $\bigoplus_{i=1}^k Ie_i$, where

$a_i \in I (1 \leq i \leq k)$. Thus

$$\varphi(a_1 e_1 + a_2 e_2 + \dots + a_k e_k) = \sum_{i=1}^k \varphi(a_i e_i) = \sum_{i=1}^k a_i \varphi(e_i).$$

Since $\varphi(e_i) \in J/I$ and $a_i \in I$, then $a_i \varphi(e_i) = 0$, so $\bigoplus_{i=1}^k Ie_i \subseteq K$. According to the Snake Lemma, $J/I \cong C$.

Consider the exact sequence

$$0 \rightarrow \frac{K}{\bigoplus_{i=1}^k Ie_i} \rightarrow \frac{\bigoplus_{i=1}^k Re_i}{\bigoplus_{i=1}^k Ie_i} \rightarrow C \rightarrow 0.$$

Since $\frac{\bigoplus_{i=1}^k Re_i}{\bigoplus_{i=1}^k Ie_i} \cong \frac{\bigoplus_{i=1}^k \bar{R}}{\bigoplus_{i=1}^k Ie_i}$ is a f.g. free \bar{R} -module and $K / (\bigoplus_{i=1}^k Ie_i)$

is a f.g. \bar{R} -module, it follows that C is a f.p. \bar{R} -module. Thus J/I is a f.p. \bar{R} -module. So R/I is a coherent ring.

Theorem 2.2 R is an S -Prüfer domain if and only if R/I is an IF ring for any f.g. S -ideal I of R .

Proof To verify the sufficient condition, we first claim that $(I^{-1})^{-1} = I$. Take $a \in I$ and $a \in S$, $\bar{I} = \frac{I}{(a)} \subseteq \frac{R}{(a)}$. By assumption, $\frac{R}{(a)}$ is an IF ring, from which we obtain that

$$\begin{aligned}
 \text{ann}_{\bar{R}}(\bar{I}) &= \{\bar{r} \in \bar{R} \mid \bar{r} \cdot \bar{I} = 0\} \\
 &= \{\bar{r} \in \bar{R} \mid rI \subseteq (a)\} \\
 &= \{\bar{r} \in \bar{R} \mid \frac{r}{a} I \subseteq R\} \\
 &= \{\bar{r} \in \bar{R} \mid \frac{r}{a} \in I^{-1}, \text{ i.e., } r \in aI^{-1}\},
 \end{aligned}$$

i.e., $\text{ann}_{\bar{R}}(\bar{I}) = \overline{aI^{-1}}$. Then:

$$\text{ann}_{\bar{R}}(\text{ann}_{\bar{R}}(\bar{I})) = \{\bar{r} \in \bar{R} \mid \bar{r} \cdot \overline{aI^{-1}} = 0\}, \text{ i.e., } raI^{-1} \subseteq aR,$$

$$\text{i.e., } rI^{-1} \subseteq R, \text{ i.e., } r \in (I^{-1})^{-1}.$$

Thus $\bar{I}_v = \text{ann}_{\bar{R}}(\text{ann}_{\bar{R}}(\bar{I})) = \bar{I}$, i.e., $\frac{(I^{-1})^{-1}}{(a)} = \frac{I}{(a)}$, i.e., $(I^{-1})^{-1} = I$.

Let Q be the quotient field of R , $(I:Q) = \{q \in Q \mid qI \subseteq I\}$. We next show that $(I:Q) = R$. Take any $q = \frac{c}{b} \in (I:Q)$, then $\frac{c}{b} \cdot I \subseteq I$. Then $cI \subseteq bI$. Since $(I:R) = I$, so $(Ib:R) = I$. Since bI is a f.g. S -ideal and $\frac{R}{Ib}$ is an IF ring. Using the double annihilator condition, we obtain that

$$Rb = (Ib :_Q (Ib :_Q Rb)) = (Ib :_Q I).$$

Since $cI \subseteq bI$, so $c \in Rb$. Thus $(I:Q) = R$.

We next show that $I = (I:II^{-1})$. It is clear that $I \subseteq (I:II^{-1})$,

so it suffices to prove that $(I:II^{-1}) \subseteq I$. Take any $a \in (I:II^{-1})$, then $aII^{-1} \subseteq I$. Thus $aI^{-1} \in R$, so $a \in (I^{-1})^{-1} = I$. Thus $(I:II^{-1}) \subseteq I$. Then $I = (I:II^{-1})$, So

$$(I:(I:II^{-1})) = (I:I) = R.$$

Using the double annihilator condition $(I:(I:II^{-1})) = II^{-1}$, we obtain that $II^{-1} = R$, then I is invertible. Then by the definition of an S -Prüfer ring, it follows that R is an S -Prüfer domain.

To prove the necessity, R/I is a coherent ring by Proposition 2.1. To prove that R/I is an IF ring, it follows from [13, Proposition 2.3] that it suffices to show that $(R/I)_{\mathfrak{m}/I}$ is an IF ring, where \mathfrak{m} is a maximal ideal of R containing I . Since I

is f.g. projective, then I_m is a projective module. Because projective modules over a local ring are free, I_m is a free ideal of R_m , and hence I_m is a principal ideal of R_m . Thus there exists $b \in I$, such that $I_m = (b)_m$, which implies that

$$\left(\frac{R}{I}\right)_m \cong \frac{R_m}{I_m} = \frac{R_m}{(b)_m}.$$

For any $\frac{R_m}{(b)_m}$ -module A , if A is a submodule of a flat $\frac{R_m}{(b)_m}$ -module, then $\frac{R_m}{(b)_m}$ is an IF ring. Note that $IA=0$, so A is also an R/I -module. Take $a \in I$ and $a \in S$; then $aA=0$. As every module can be expressed as the homomorphic image of a free module, we have the exact sequence:

$$0 \longrightarrow L \longrightarrow F \xrightarrow{\psi} A \longrightarrow 0.$$

where F is free, $L = \text{Ker } \psi$. Since

$$\psi(aF) = a\psi(F) = aA = 0,$$

so $aF \subseteq L$, thus L is an S -module. Moreover, R is an S -Prüfer domain, by [1, Theorem 2.4], the S -module L is flat. Now consider the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & A \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & L & \rightarrow & F & \xrightarrow{\psi} & A \rightarrow 0 \\ & & \downarrow \cdot b & & \downarrow \cdot b & & \downarrow \cdot b \\ 0 & \rightarrow & L & \rightarrow & F & \xrightarrow{\psi} & A \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \frac{L}{bL} & \rightarrow & \frac{F}{bF} & \longrightarrow & \frac{A}{bA} \rightarrow 0. \end{array}$$

Since $b \in I$, $IA=0$, so $bA=0$, Then $\frac{A}{bA} = A$. Applying the Snake Lemma, one obtains the subsequent exact sequence:

$$0 \rightarrow A \rightarrow \frac{L}{bL} \rightarrow \frac{F}{bF} \rightarrow A \rightarrow 0.$$

Since

$$\frac{L}{bL} \cong L \otimes_R R/(b),$$

and L is a flat R -module, then L/bL is a flat $R/(b)$ -module. Thus $\left(\frac{L}{bL}\right)_m$ is a flat $\left(\frac{R}{(b)}\right)_m$ -module. Since localization preserves exactness, we obtain that

$$0 \rightarrow A_m \rightarrow \left(\frac{L}{bL}\right)_m$$

is exact. Thus A_m is a submodule of a flat $\frac{R_m}{bR_m}$ -module $\left(\frac{L}{bL}\right)_m$.

We next show that $A_m = A$. Since A is an $\left(\frac{R}{I}\right)_m$ -module, so $A\left(\frac{R}{I}\right)_m = A$. Moreover, since $A\left(\frac{R}{I}\right)_m = A_m$, it follows that $A_m = A$. Therefore, $\frac{R_m}{bR_m}$ is an IF ring. Hence R/I is an IF ring.

Let R be a domain and I an ideal of R . Define $I^{-1} = \{x \in K \mid xI \subseteq R\}$ and $I_v = (I^{-1})^{-1}$, where K is the quotient field of R . For a domain R , an ideal I is called a v -ideal of R if $I_v = I$. Since f.g. projective modules are f.p. ([6, Example 3.4.2]), so every f.g. S -ideal of S -Prüfer domains is f.p..

Lemma 2.3 If every finitely S -ideal I of a domain R is f.p., then I^{-1} is also f.g..

Proof Let I is a f.g. S -ideal of R . We may assume that $I = (a_1, a_2, \dots, a_{n-1}, a_n)$, where $a_1 \in S$. We proceed by mathematical induction on n . For $n=1$, let $I = a_1R$. Then $I^{-1} = \frac{1}{a_1}R$, so I^{-1} generated by $\frac{1}{a_1}$ is a fractional ideal of R , i.e., I^{-1} is f.g..

Now assume that $(a_1, a_2, \dots, a_{n-1})^{-1}$ is a f.g. R -module. Note that

$$I = (a_1, a_2, \dots, a_{n-1}, a_n) = (a_1, a_2, \dots, a_{n-1}) + (a_n) = I' + a_nR.$$

Then

$$I^{-1} = (I' + a_nR)^{-1} = (I')^{-1} \cap (a_nR)^{-1}.$$

Since $(a_nR)^{-1} = \frac{1}{a_n}R$, so

$$(I')^{-1} + \frac{1}{a_n}R \cong a_1 a_n \left((I')^{-1} + \frac{1}{a_n}R \right) = a_1 a_n (I')^{-1} + a_1 R.$$

Since $a_1 a_n (I')^{-1} + a_1 R$ is a f.g. S -ideal, then $(I')^{-1} + (a_nR)^{-1}$ is f.p.. From the exact sequence $0 \rightarrow (I')^{-1} \cap (a_nR)^{-1} \rightarrow (I')^{-1} \oplus (a_nR)^{-1} \rightarrow (I')^{-1} + (a_nR)^{-1} \rightarrow 0$, and $(I')^{-1} \oplus (a_nR)^{-1}$ is f.g. R -module, so $(I')^{-1} \cap (a_nR)^{-1}$ is f.g.. Thus I^{-1} is f.g..

Theorem 2.4 R is an S -Prüfer domain if and only if R is integrally closed in R_S , every f.g. S -ideal I of R is f.p., and every f.g. S -ideal of R is a v -ideal.

Proof To prove necessity, we know that If R is an S -Prüfer domain, then every f.g. S -ideal of R is projective, hence f.p..

We next show that R is integrally closed in R_S . That is, we prove that if $\alpha \in R_S$ is integral over R , then $\alpha \in R$. Assume $\alpha = \frac{a}{s}$ is integral over R , where $a \in R$, $s \in S$. Then there exists a positive integer n such that

$$\alpha^n + r_1 \alpha^{n-1} + \dots + r_{n-1} \alpha + r_n = 0,$$

where $r_i \in R$, $i=1, 2, \dots, n$. Let

$$J = (\alpha^{n-1}, \alpha^{n-2}, \dots, \alpha, 1) = R\alpha^{n-1} + R\alpha^{n-2} + \dots + Ra + R.$$

It is straightforward to verify that $J \cong s^{n-1}J$. Denote $L = s^{n-1}J$, then L is a f.g. ideal of R . Since $1 \in J$, then $s^{n-1} \cdot 1 \in s^{n-1}J$. Hence L is f.g. S -ideal. Since R is an S -Prüfer domain, so L is invertible, that is, $LL^{-1} = R$. Note that $aJ \subseteq J$, so $\alpha s^{n-1}J \subseteq s^{n-1}J$, i.e., $\alpha L \subseteq L$. Then $rLL^{-1} \subseteq LL^{-1}$, $\alpha R \subseteq R$. Thus $\alpha \in R$, so R is integrally closed in R_S .

We next show that any f.g. S -ideal I of R is a v -ideal. Since $I^{-1} \subseteq R$, we have $I \subseteq I_v$. Let $x \in I_v$, then $xI^{-1} \subseteq R$, which implies $xI^{-1}I \subseteq I$. Since I is invertible, we have $II^{-1} = R$, so that $x \in I$, i.e., $I_v \subseteq I$. Therefore $I = I_v$, and hence I is a v -ideal of R .

To prove sufficiency, let I be a f.g. S -ideal of R . Then there exists an $s \in S$ such that $s \in I$. Since

$$I = IR \subseteq II^{-1} \subseteq R,$$

So $s \in II^{-1}$, it follows that II^{-1} is an S -ideal of R . Since I is f.g., I^{-1} is also f.g. by Lemma 2.3, so II^{-1} is f.g.. Hence II^{-1} is a f.g. S -ideal, and therefore a v -ideal, that is, $(II^{-1})_v = II^{-1}$.

Let $r \in (II^{-1})^{-1}$. Then $rII^{-1} \subseteq R$, $rs \in R$, and $r \in R_S$. Since $(rI)I^{-1} \subseteq R$, we have $rI \subseteq (I^{-1})^{-1} = I$. Hence r is integral over R . Since R is integrally closed in R_S , it follows that $r \in R$. Then $(II^{-1})^{-1} \subseteq R$. Clearly $R \subseteq (II^{-1})^{-1}$, so $(II^{-1})^{-1} = R$. Thus $II^{-1} = (II^{-1})_v = R^{-1} = R$, hence I is invertible, and R is an S -Prüfer domain.

3. Equivalent Characterizations of

$$\sum_{i=1}^n (a_i \otimes b_i) = 0$$

It is a classical conclusion that an domain R is Prüfer whenever each (f.g.) ideal of R is a flat module. R is an S -Prüfer domain if and only if each (f.g.) S -ideal is flat [1, Theorem 4.1.3]. The definitions of flat modules and generalized flat modules both rely on one important functor in homological algebra: tensor product \otimes . We now give an

equivalent characterization of $\sum_{i=1}^n (a_i \otimes b_i) = 0$ in an R -module $A \otimes_R B$.

Theorem 3.1 Let A and B be R -modules, then $\sum_{i=1}^n (a_i \otimes b_i) = 0 (a_i \in A, b_i \in B)$ if and only if there exist $c_j \in B (j=1, 2, \dots, m)$, and a positive integer $n' \geq n, a_{n+1}, \dots, a_{n'} \in A$, and $r_{ij} \in R (i=1, 2, \dots, n'; j=1, 2, \dots, m)$ such that

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \\ r_{n+1,1} & r_{n+1,2} & \cdots & r_{n+1,m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n',1} & r_{n',2} & \cdots & r_{n',m} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \quad (1)$$

and

$$(a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{n'}) = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ r_{21} & r_{22} & \cdots & r_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nm} \\ r_{n+1,1} & r_{n+1,2} & \cdots & r_{n+1,m} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n',1} & r_{n',2} & \cdots & r_{n',m} \end{pmatrix} = (0, 0, \dots, 0). \quad (2)$$

Proof To prove necessity, we assume that $\sum_{i=1}^n (a_i \otimes b_i) = 0 (a_i \in A, b_i \in B)$ in $A \otimes_R B$. For an R -module A , there exist a free module $F = \bigoplus_{k \in I} R x_k$ and an epimorphism $\phi: F \rightarrow A$ such that $\phi(x_i) = a_i (i=1, 2, \dots, n)$ by [5, Theorem 1.6.7], thus inducing an exact sequence $0 \rightarrow K \xrightarrow{\lambda} F \xrightarrow{\phi} A \rightarrow 0$, where $K = \text{Ker}(\phi)$. Thus we have the exact sequence

$$K \otimes B \xrightarrow{\lambda \otimes 1_B} F \otimes B \xrightarrow{\phi \otimes 1_B} A \otimes B \rightarrow 0.$$

$$\text{Note } (\phi \otimes 1_B) \left(\sum_{i=1}^n (x_i \otimes b_i) \right) = \sum_{i=1}^n (\phi(x_i) \otimes b_i) = \sum_{i=1}^n (a_i \otimes b_i) = 0,$$

so

$$\sum_{i=1}^n (x_i \otimes b_i) \in \text{Ker}(\phi \otimes 1_B) \subseteq K \otimes B.$$

Thus exists $\sum_{j=1}^m (y_j \otimes c_j) \in K \otimes B (y_j \in K, c_j \in B)$, such that

$$(\lambda \otimes 1_B) \left(\sum_{j=1}^m (y_j \otimes c_j) \right) = \sum_{j=1}^m (\lambda(y_j) \otimes c_j) = \sum_{j=1}^m (y_j \otimes c_j) = \sum_{i=1}^n (x_i \otimes b_i).$$

From $y_j \in K \subseteq F$ follows that $y_j = \sum_{i=1}^{n'} r_{ij} x_i$, where $n' \geq n$. Hence

$$\sum_{j=1}^m (y_j \otimes c_j) = \sum_{j=1}^m \left(\sum_{i=1}^{n'} (r_{ij} x_i \otimes c_j) \right) = \sum_{i=1}^{n'} \left(x_i \otimes \left(\sum_{j=1}^m r_{ij} c_j \right) \right).$$

Since we have $\sum_{j=1}^m (y_j \otimes c_j) = \sum_{i=1}^n (x_i \otimes b_i)$ in $F \otimes_R B$, it

follows that $b_i = \sum_{j=1}^m r_{ij} c_j, i=1, 2, \dots, n$ and

$\sum_{j=1}^m r_{ij} c_j = 0, i=n+1, n+2, \dots, n'$, that is, Equation(1) holds. Since

$$\phi(y_i) = \phi \left(\sum_{j=1}^m r_{ij} x_j \right) = \phi \left(\sum_{i=1}^n r_{ij} x_i + \sum_{i=n+1}^{n'} r_{ij} x_i \right) = \sum_{i=1}^n r_{ij} a_i + \sum_{i=n+1}^{n'} r_{ij} \phi(x_i).$$

Let $a_i = \phi(x_i)$, where $i=n+1, n+2, \dots, n'$. So

$$\phi(y_i) = \sum_{i=1}^{n'} r_{ij} a_i = 0,$$

$j=1, 2, \dots, m$, that is, Equation (2) holds.

To prove sufficiency,

$$\begin{aligned} \sum_{i=1}^n (a_i \otimes b_i) &= \sum_{i=1}^n (a_i \otimes b_i) + \sum_{i=n+1}^{n'} (a_i \otimes 0) = \sum_{i=1}^n \left(a_i \otimes \left(\sum_{j=1}^m r_{ij} c_j \right) \right) + \sum_{i=n+1}^{n'} \left(a_i \otimes \left(\sum_{j=1}^m r_{ij} c_j \right) \right) \\ &= \sum_{j=1}^m r_{ij} \left(\sum_{i=1}^n a_i \otimes c_j \right) + \sum_{j=1}^m r_{ij} \left(\sum_{i=n+1}^{n'} a_i \otimes c_j \right) = \sum_{j=1}^m r_{ij} \left(\sum_{i=1}^n a_i \otimes c_j \right) \\ &= \sum_{j=1}^m \left(\left(\sum_{i=1}^n a_i r_{ij} \right) \otimes c_j \right) \\ &= \sum_{j=1}^m (0 \otimes c_j) = 0. \end{aligned}$$

In fact, let A and B be R -modules. If for $a_i \in A, b_i \in B, i=1, 2, \dots, n$, there exist $c_j \in B (j=1, 2, \dots, m)$ and $r_{ij} \in R (i=1, 2, \dots, n; j=1, 2, \dots, m)$, such that $b_i = \sum_{j=1}^m r_{ij} c_j$ for any i and $\sum_{i=1}^n a_i r_{ij} = 0$ for any j . Then $\sum_{i=1}^n (a_i \otimes b_i) = 0$. However, the converse is not true.

We now give an example to show that the converse fails.

Example 3.2 Let R be a ring Z , where Z denote the ring of integers, so $A=Z, B=Z/3Z$ are R -modules. In $Z \otimes_Z Z/3Z$, we have $3 \otimes \bar{1} = 0$. If there exist $c_i \in B, r_i \in R (i=1, 2, \dots, m)$ such that

$$\bar{1} = \sum_{i=1}^m r_i c_i = r_1 c_1 + r_2 c_2 + \dots + r_m c_m \text{ and } 3r_1 = 3r_2 = \dots = 3r_m = 0.$$

Thus $r_1 = r_2 = \dots = r_m = 0$, so $\bar{1} = \bar{0}$, a contradiction.

Note: Example 3.2 shows that for $3 \otimes \bar{1} = 0$ in $Z \otimes_Z Z/3Z$, there do not exist $c_i \in B, r_i \in R (i=1, 2, \dots, m)$, such that $\bar{1} = \sum_{i=1}^m r_i c_i$ and $3r_1 = 3r_2 = \dots = 3r_m = 0$. By Theorem 3.1, there exist $c_j \in B (j=1, 2, \dots, m)$, a positive integer $n' \geq 1$, $a_2, \dots, a_{n'} \in A$, and $r_{ij} \in R (i=1, 2, \dots, n'; j=1, 2, \dots, m)$, such that

$$\begin{pmatrix} \bar{1} \\ \bar{0} \\ \vdots \\ \bar{0} \end{pmatrix} = \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ r_{21} & \cdots & r_{2m} \\ \vdots & \ddots & \vdots \\ r_{n',1} & \cdots & r_{n',m} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$$

and

$$(a_1, a_2, \dots, a_{n'}) = \begin{pmatrix} r_{11} & \cdots & r_{1m} \\ r_{21} & \cdots & r_{2m} \\ \vdots & \ddots & \vdots \\ r_{n',1} & \cdots & r_{n',m} \end{pmatrix} = (0, 0, \dots, 0).$$

In fact, we can take $n'=2, m=1, c_1 = \bar{1}, r_{11} = -2, r_{21} = 3, a_2 = 2$, satisfying the above conditions, that is,

$$\begin{pmatrix} \bar{1} \\ \bar{0} \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \bar{1}$$

and

$$(3 \ 2) \begin{pmatrix} -2 \\ 3 \end{pmatrix} = 0.$$

For an R -module A , if A is f.g., we have the following corollary.

Corollary 3.3 Let A and B be R -modules. If $A = Ra_1 + Ra_2 + \dots + Ra_n$, where $a_i \in A (i=1, 2, \dots, n)$. Then

$$\sum_{i=1}^n (a_i \otimes b_i) = 0 (b_i \in B);$$

if and only if there exist $c_j \in B (j=1, 2, \dots, m)$ and

$r_{ij} \in R(i=1,2, \dots, n; j=1,2, \dots, m)$ such that $b_i = \sum_{j=1}^m r_{ij}c_j$ for any i , and $\sum_{i=1}^n a_i r_{ij} = 0$ for any j .

Proof To prove necessity, we assume that $\sum_{i=1}^n (a_i \otimes b_i) = 0$, where $A = Ra_1 + Ra_2 + \dots + Ra_n$ is f.g., $b_i \in B$. For an R -module A , there exist a free module $F = \bigoplus_{k=1}^n Rx_k$ and an epimorphism $\phi: F \rightarrow A$ satisfying $\phi(x_i) = a_i (i=1,2, \dots, n)$ by [5, Theorem 1.6.7]. Thus inducing an exact sequence

$$0 \rightarrow K \xrightarrow{\lambda} F \xrightarrow{\phi} A \rightarrow 0, \text{ where } K = \text{Ker}(\phi).$$

Thus we have the exact sequence

$$K \otimes B \xrightarrow{\lambda \otimes 1_B} F \otimes B \xrightarrow{\phi \otimes 1_B} A \otimes B \rightarrow 0.$$

Note $(\phi \otimes 1_B)(\sum_{i=1}^n (x_i \otimes b_i)) = \sum_{i=1}^n (\phi(x_i) \otimes b_i) = \sum_{i=1}^n (a_i \otimes b_i) = 0$, so

$$\sum_{i=1}^n (x_i \otimes b_i) \in \text{Ker}(\phi \otimes 1_B) \subseteq K \otimes B.$$

Thus, there exists $\sum_{j=1}^m (y_j \otimes c_j) \in K \otimes B (y_j \in K, c_j \in B)$ such that:

$$(\lambda \otimes 1_B) \left(\sum_{j=1}^m (y_j \otimes c_j) \right) = \sum_{j=1}^m (\lambda(y_j) \otimes c_j) = \sum_{j=1}^m (y_j \otimes c_j) = \sum_{i=1}^n (x_i \otimes b_i).$$

From $y_j \in K \subseteq F$, we have $y_j = \sum_{i=1}^n r_{ij}x_i$. Therefore

$$\sum_{j=1}^m (y_j \otimes c_j) = \sum_{j=1}^m \left(\sum_{i=1}^n (r_{ij}x_i \otimes c_j) \right) = \sum_{i=1}^n \left(x_i \otimes \left(\sum_{j=1}^m r_{ij}c_j \right) \right).$$

Since $\sum_{j=1}^m y_j \otimes c_j = \sum_{i=1}^n x_i \otimes b_i$, we obtain

$$b_i = \sum_{j=1}^m r_{ij}c_j, i=1,2, \dots, n \text{ and } \phi(y_i) = \phi\left(\sum_{j=1}^m r_{ij}x_j\right) = \sum_{j=1}^m r_{ij}a_j = 0.$$

To prove sufficiency,

$$\begin{aligned} \sum_{i=1}^n (a_i \otimes b_i) &= \sum_{i=1}^n \left(a_i \otimes \left(\sum_{j=1}^m r_{ij}c_j \right) \right) = \sum_{j=1}^m r_{ij} \left(\sum_{i=1}^n a_i \otimes c_j \right) \\ &= \sum_{j=1}^m \left(\left(\sum_{i=1}^n a_i r_{ij} \right) \otimes c_j \right) = \sum_{j=1}^m (0 \otimes c_j) \\ &= 0. \end{aligned}$$

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