

Continuous Lower Bounds for Moments of the Mixed Product of Twisted L -functions

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Abstract: Moments of central values in family of L -functions are important subjects in analytic number theory. The twisted L -functions is an important class of automorphic L -functions. Recently, using the lower bound principle of Heap and Soundararajan, Chen et al. have obtained the sharp lower bounds for all positive real k -th ($k > 1$) moments of the mixed product of two twisted L -functions at the central value, when there exists a divisor q_0 such that $q_0 | q$ and $q^\eta \ll q_0 \ll q^{1/2-\eta}$. Here q is the modulus of Dirichlet character and η is a small real number. Based on Chen et al. and Blomer et al. the work of on the first moment of the mixed product of two twisted L -function at the central value, when the modulus q of the Dirichlet character is prime, in this paper, we study lower bounds for all positive real k -th ($k > 1$) moments of the mixed product of two distinct twisted L -function at the central value for the prime modulus case and obtain the sharp lower bounds.

Keywords: Twisted L -functions; Cuspidal Holomorphic Hecke Eigenform; Moments Lower Bounds.

1. Introduction

In recent years, the study of mean value estimates for central values of L -functions has attracted widespread attention and led to significant progress. Based on important works by Keating-Snaith [10, 11], Conrey, Farmer, Keating, and Rubinstein-Snaith [4], conjectured asymptotic formulas for the mean values of certain L -functions at their central points have now been established. However, the asymptotic formula of the conjecture has only been verified for low-order means. For the majority of L -functions, even the upper or lower bounds of the correct order remain unestablished. Rudnick-Soundararajan [15, 16] developed a highly effective method for providing lower bounds of the correct order for all positive rational moments (> 1) of certain families of L -functions at their central values.

In 2022, Heap-Soundararajan et al. [8] employed short Dirichlet polynomials to approximate the Euler product of powers of the zeta function, thereby successfully obtaining the correct order of lower bounds for all positive real-power mean values of the zeta function along the critical line.

Let f be a cuspidal holomorphic Hecke eigenform of even weight κ for the modular group $SL_2(\mathbb{Z})$. Then f has the following Fourier expansion at the cusp ∞

$$f(z) = \sum_{n \geq 1} \lambda_f(n) n^{\frac{\kappa-1}{2}} e(nz),$$

for $\Im(z) > 0$ with normalized $\lambda(1) = 1$.

Deligne's bound [6] gives

$$|\lambda(n)| \leq d(n),$$

where $d(n)$ denotes the number of positive divisors of n .

The twisted L -function associated to f twisted by a primitive Dirichlet character modulo q ($q \not\equiv 2 \pmod{4}$) is given by

$$L(s, f \otimes \chi) = \sum_{n \geq 1} \frac{\lambda_f(n) \chi(n)}{n^s}.$$

In this paper, we focus on establishing continuous lower bounds for the mean value of the mixed product of twisted L -

functions at their central points. In 2025, Chen et al. successfully established strong lower bound estimates for the continuous moments of the mixed product of twisted L -functions at their central points, utilizing the method developed by Heap and Soundararajan.

This paper investigates strong lower bounds for continuous higher-order mean estimates of mixed products of two twisted L -functions at central values in the case of prime modulus q .

Theorem 1. Let f_1, f_2 be two fixed distinct cuspidal holomorphic Hecke eigenforms for the modular group $SL_2(\mathbb{Z})$ and respective even weight κ_1, κ_2 with $\kappa_1 \equiv \kappa_2 \pmod{4}$. Let $q \not\equiv 2 \pmod{4}$, then for any positive real number $k > 1$ and sufficiently large prime modulus q , we have

$$\sum_{\chi \pmod{q}}^* |L(1/2, f_1 \otimes \chi) \overline{L(1/2, f_2 \otimes \chi)}|^k \gg_k q(\log q)^{\frac{k^2}{2}},$$

where the asterisk restricts the summation to all primitive characters modulo q .

In this paper, we adopt the following conventions. The modulus q of the Dirichlet character is a sufficiently large prime. $A \asymp B$ means $A \ll B$ and $B \ll A$. Let ε be an arbitrarily small positive number.

2. Set up and Outline of the Proof

Let N_0, M_0 be two large natural numbers depending at most on k and η . Suppose that $\{l_j\}_{1 \leq j \leq R}$ is a sequence of even natural numbers, where $l_1 = 2[N_0 \log \log q]$ and $l_{j+1} = 2[N_0 \log l_j]$ for $j \geq 1$. Write R to be the largest natural number such that $l_R > 10^{M_0}$. We will choose M_0 large enough so that $l_{j-1} > l_j^2$ for all $2 \leq j \leq R$. Then we have

$$R \ll \log \log l_1, \sum_{j=1}^R \frac{1}{l_j} \leq \frac{2}{l_R}.$$

Let P_1 be the set of primes in the interval $(2, q^{1/l_1}^2]$, and P_j be the set of primes in the interval $(q^{1/l_{j-1}^2}, q^{1/l_j^2}]$ with $2 \leq j \leq R$. For each $1 \leq j \leq R$, we define

$$\mathcal{P}_j(\chi) = \sum_{p \in P_j} \frac{\lambda_1(p)\chi(p) + \lambda_2(p)\bar{\chi}(p)}{\sqrt{p}}, \quad (1)$$

$$E_{l_j}(\chi) = \sum_{h=0}^{l_j} \frac{x^h}{h!} \mathcal{N}_j(\chi, \alpha) = E_{l_j}(\alpha \mathcal{P}_j(\chi)) \quad (2)$$

with $\alpha \in \mathbb{R}$.

Let $\Omega(n)$ denote the number of distinct prime factors of n , and $\nu_j(n)$ be a completely multiplicative function and $\nu_j(p) = \lambda_j(p)$ for $j=1,2$. Suppose that $g(n)$ is a multiplicative function and $g(p^r) = r!$ with $r \in \mathbb{N}$. Recalling the definition (2.1) of $\mathcal{N}_j(\chi, \alpha)$ with $1 \leq j \leq R$, we have

$$\mathcal{N}_j(\chi, \alpha) = \sum_{\substack{m_j, n_j \in S(P_j) \\ \Omega(m_j, n_j) \leq l_j}} \frac{\alpha^{\Omega(m_j, n_j)} \nu_1(m_j) \nu_2(n_j) \chi(m_j) \bar{\chi}(n_j)}{\sqrt{m_j n_j} g(m_j) g(n_j)},$$

where $S(P_j) = \{n \in \mathbb{N} : p | n \Rightarrow p \in P_j\}$. Therefore,

$$\begin{aligned} \mathcal{N}(\chi, \alpha) &= \prod_{j=1}^R \mathcal{N}_j(\chi, \alpha) \\ &= \sum_{m, n \in \mathbb{N}} \sum_{\Omega(m_j, n_j) \leq l_j, 1 \leq j \leq R} \frac{\alpha^{\Omega(mn)} \nu_1(m) \nu_2(n) \chi(m) \bar{\chi}(n)}{\sqrt{mn} g(m) g(n)}, \quad (3) \end{aligned}$$

where

$$\mathcal{N} := \{n \in \mathbb{N} : n = n_1 n_2 \cdots n_R, n_j \in S(P_j), 1 \leq j \leq R\}.$$

To prove Theorem 1, it suffices to prove

$$\begin{aligned} &\sum_{\chi(\bmod q)}^* |L(1/2, f_1 \otimes \chi) \overline{L(1/2, f_2 \otimes \chi)}|^k \gg_k q(\log q)^{\frac{k^2}{2}}, \\ &\text{for } k > 1/2. \text{ By the Hölder inequality, we have} \\ &\sum_{\chi(\bmod q)}^* L(1/2, f_1 \otimes \chi) \overline{L(1/2, f_2 \otimes \chi)} \mathcal{N}(\chi, k-1) \mathcal{N}(\bar{\chi}, k) \\ &\leq \left(\sum_{\chi(\bmod q)}^* |L(1/2, f_1 \otimes \chi) \overline{L(1/2, f_2 \otimes \chi)}|^{2k} \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{\chi(\bmod q)}^* |\mathcal{N}(\chi, k-1) \mathcal{N}(\bar{\chi}, k)|^{\frac{2k}{2k-1}} \right)^{\frac{2k-1}{2k}}. \quad (4) \end{aligned}$$

Hence, Theorem 1 follows directly from the above equation and the following proposition.

Proposition 1. With above notations, we have

$$\begin{aligned} &\sum_{\chi(\bmod q)}^* L(1/2, f_1 \otimes \chi) \overline{L(1/2, f_2 \otimes \chi)} \mathcal{N}(\chi, k-1) \mathcal{N}(\bar{\chi}, k) \\ &\gg_k q(\log q)^{2k^2}. \end{aligned}$$

Proposition 2. With above notations, we have

$$\sum_{\chi(\bmod q)}^* |\mathcal{N}(\chi, k-1) \mathcal{N}(\bar{\chi}, k)|^{\frac{2k}{2k-1}} \ll_k q(\log q)^{2k^2}.$$

3. Prerequisite Knowledge

In order to proof the above Propositions, we need some necessary lemmas. For the readers' convenience, we collect them below.

Lemma 3. [3, Lemma 1.6] There exists a smooth non-negative function $W(x)$ supported on $[1/2, 2]$ and satisfying

$$W^j(x) \ll_j 1$$

for any $j \geq 0$, such that

$$\sum_{\nu \geq 0} W\left(\frac{x}{2^\nu}\right) = 1$$

for any $x \geq 1$.

Lemma 4. [5, Lemma 2] (Approximate functional equation) We have

$$\begin{aligned} &L(1/2, f_1 \otimes \chi) \overline{L(1/2, f_2 \otimes \chi)} \\ &= \sum_{m, n} \frac{\lambda_1(m) \lambda_2(n) (\chi(m) \bar{\chi}(n) + \bar{\chi}(m) \chi(n))}{\sqrt{mn}} V\left(\frac{mn}{q^2}\right), \end{aligned}$$

where

$$V(x) = \frac{1}{2\pi i} \int_{(2)} \frac{\Gamma(\kappa_1/2 + s) \Gamma(\kappa_2/2 + s)}{(2\pi)^{2s} \Gamma(\kappa_1/2) \Gamma(\kappa_2/2)} x^{-s} ds$$

satisfies $V^j(x) \ll_{j,A} (1+x)^{-A}$ for all $A, j \geq 0$.

Lemma 5. [3, Lemma 6] (Voronoi summation formula) Let $c \in \mathbb{N}, b \in \mathbb{Z}$ and assume $(b, c) > 1$. Let H be a smooth compactly supported function, and let $N > 0$. Let $\lambda(n)$ denote the normalized Hecke eigenvalues of a Hecke eigenform of level 1 and weight κ . Then

$$\sum_n \lambda(n) e\left(\frac{bn}{c}\right) H\left(\frac{n}{N}\right) = \frac{N}{c} \sum_n \lambda(n) e\left(-\frac{\bar{b}n}{c}\right) \mathcal{H}\left(\frac{n}{c^2/N}\right),$$

where

$$\mathcal{H}(y) = 2\pi i^\kappa \int_0^\infty H(x) J_{\kappa-1}(4\pi\sqrt{xy}) dx.$$

Let $l_1, l_2, d, q, h \in \mathbb{N}, N, M \geq 1$ and define

$$D(l_1, l_2, h, N, M) = \sum_r \lambda_1(m) \lambda_2(n) \overline{W}_1\left(\frac{l_1 m}{M}\right) \overline{W}_2\left(\frac{l_2 n}{N}\right)$$

and

$$S(l_1, l_2, d, N, M) = \sum_r D(l_1, l_2, rd, N, M).$$

where \overline{W}_1 and \overline{W}_2 are compactly supported on $[1/2, 2]$ and satisfy $\overline{W}_i^{(j)} \ll q^{j\varepsilon}$ with $1 \leq i \leq 2, j \geq 0$.

Lemma 6. [3, Proposition 8] With above notations, assume that $N \geq 20M$, then we have

$$\begin{aligned} &S(l_1, l_2, d, N, M) \\ &\ll (dM)^\varepsilon \left(\frac{N}{d^{1/2}} + \frac{N^{5/4} M^{1/4}}{d} + \frac{N^{3/4} M^{1/4}}{d^{1/4}} + \frac{NM^{1/2}}{d^{3/4}} \right). \end{aligned}$$

Lemma 7. [3, Lemma 5] Let f_1, f_2 be two distinct cuspidal holomorphic Hecke eigenforms of even weight κ_1, κ_2 for the modular group $SL_2(\mathbb{Z})$ with normalized Fourier coefficients λ_1, λ_2 respectively, then we have

$$\sum_{p \leq x} \frac{\lambda_i^2(p)}{p} = \log \log x + O(1), i=1, 2$$

and

$$\sum_{p \leq x} \frac{\lambda_1(p) \lambda_2(p)}{p} = O(1).$$

Lemma 8. [13, Lemma 1.1] Let $\alpha = (\alpha_m)_{m \leq M}$ and $\beta = (\beta_n)_{n \leq N}$ be sequences of complex numbers. Let q be a prime. $(ac, q) = 1$, Let M and N be real numbers and $M \asymp m, N \asymp n$ and satisfying

$$1 \leq M \leq Nq^{1/4}, q^{1/4} < MN \leq q^{5/4},$$

then we have

$$\begin{aligned} &\sum_m \sum_n \alpha_m \beta_n Kl_2(ac; q) \\ &\ll q^\varepsilon \|\alpha\|_2 \|\beta\|_2 (MN)^{1/2} (M^{-1/2} + (MN)^{-3/16} q^{1/64}) \end{aligned}$$

and

$$Kl_2(ac; q) = \frac{1}{\sqrt{q}} S(a, 1, q) = \frac{1}{\sqrt{q}} \sum_{\substack{x(\bmod q) \\ (x, q)=1}} e\left(\frac{mx + n\bar{x}}{q}\right).$$

4. The Proof of Proposition 1

By Lemma 4 and the elementary identity [9, p.46 (3.8)]

$$\sum_{\chi(\bmod q)}^* \chi(a) = \sum_{d|(a-1, q)} \phi(d) \mu\left(\frac{q}{d}\right), (a, q) = 1, \quad (5)$$

we get

$$\sum_{\chi(\bmod q)}^* L(1/2, f_1 \otimes \chi) \overline{L(1/2, f_2 \otimes \chi)} \mathcal{N}(\chi, k-1) \mathcal{N}(\bar{\chi}, k) = S_1 + S_2 \quad (6)$$

where

$$S_1 = \sum_{d|q} \phi(d) \mu\left(\frac{q}{d}\right) \sum_{a, b, c, \delta \leq q^{2/r}} \frac{(k-1)^{\Omega(ab)} k^{\Omega(c\delta)} v_1(ac) v_2(b\delta)}{\sqrt{abc\delta} g(a) g(b) g(c) g(\delta)} \\ \times \sum_{\substack{(mn, q)=1 \\ ma\delta \equiv nbc(\bmod d)}} \frac{\lambda_1(m) \lambda_2(n)}{\sqrt{mn}} V\left(\frac{mn}{q^2}\right)$$

and

$$S_2 = \sum_{d|q} \phi(d) \mu\left(\frac{q}{d}\right) \sum_{a, b, c, \delta \leq q^{2/r}} \frac{(k-1)^{\Omega(ab)} k^{\Omega(c\delta)} v_1(ac) v_2(b\delta)}{\sqrt{abc\delta} g(a) g(b) g(c) g(\delta)} \\ \times \sum_{\substack{(mn, q)=1 \\ ma\delta \equiv nbc(\bmod d)}} \frac{\lambda_1(m) \lambda_2(n)}{\sqrt{mn}} V\left(\frac{mn}{q^2}\right),$$

without causing confusion, we have omitted the constraints on variable a, b, c, δ in equation (3). In this paper, only item S_1 is addressed, because the estimation for item S_2 is similar to that for S_1 . The following decomposition is performed on S_1 ,

$$S_1 = D_1 + E_1, \quad (7)$$

where D_1 and E_1 denote the diagonal term ($ma\delta = nbc$) and non-diagonal term ($ma\delta \neq nbc$) respectively.

4.1. Compute the Contribution of D_1

The treatment regarding D_1 is similar to that in paper [18], as this case is a specific instance of the scenario discussed in [18]. For the sake of brevity, further details are omitted here. Consequently,

$$D_1 \gg q(\log q)^{2k^2}. \quad (8)$$

4.2. Compute the Contribution of E_1

Next, we will estimate E_1 using the method described in reference [3]. Without causing confusion, we omit the constraints on variable a, b, c, δ in Equation (3). Using trivial estimates, we have

$$E_1 \ll q^\varepsilon \sum_{d|q} \phi(d) \\ \times \sum_{\substack{a, b, c, \delta \leq q^{\eta/200} \\ (abc\delta, q)=1}} \frac{1}{\sqrt{abc\delta}} \left| \sum_{\substack{(mn, q)=1 \\ ma\delta \equiv nbc(\bmod d) \\ ma\delta \neq nbc}} \frac{\lambda_1(m) \lambda_2(n)}{\sqrt{mn}} V\left(\frac{mn}{q^2}\right) \right|, \quad (9)$$

where $a, b, c, \delta \leq q^{\eta/200}$ is available because we can take a sufficiently large M_0 for a fixed $\eta (0 < \eta < 1/2)$. Next, we conduct a detailed analysis of the inner sum in the above expression.

By the Mellin inversion, $\mathcal{V}(s)$ denotes the Mellin transform of $V(x)$. Combining Lemma 3 and Lemma 4, the

inner summation in the above expression can be rewritten as

$$\sum_{\substack{(mn, q)=1 \\ ma\delta \equiv nbc(\bmod q) \\ ma\delta \neq nbc}} \frac{\lambda_1(m) \lambda_2(n)}{\sqrt{mn}} V\left(\frac{mn}{q^2}\right) \\ = \frac{1}{2\pi i} \int_{(\varepsilon)} \mathcal{V}(s) \sum_{\substack{(mn, q)=1 \\ ma\delta \equiv nbc(\bmod d) \\ ma\delta \neq nbc}} \frac{\lambda_1(m) \lambda_2(n)}{(mn)^{1/2+s}} q^{2s} \frac{ds}{s} \\ = \frac{1}{2\pi i} \sum_{\substack{M=2^{k_1}, N=2^{k_2} \\ k_1, k_2 \geq 0 \\ MN \leq q^{2+\varepsilon}}} \int_{(\varepsilon)} \mathcal{V}(s) \\ \times \sum_{\substack{(mn, q)=1 \\ ma\delta \equiv nbc(\bmod d) \\ ma\delta \neq nbc}} \frac{\lambda_1(m) \lambda_2(n)}{(mn)^{1/2+s}} W_1\left(\frac{m}{M}\right) W_2\left(\frac{n}{N}\right) q^{2s} \frac{ds}{s} \\ = \frac{1}{2\pi i} \sum_{\substack{M=2^{k_1}, N=2^{k_2} \\ k_1, k_2 \geq 0 \\ MN \leq q^{2+\varepsilon}}} \int_{(\varepsilon)} \mathcal{V}(s) \\ \times \sum_{\substack{(mn, q)=1 \\ ma\delta \equiv nbc(\bmod d) \\ ma\delta \neq nbc}} \frac{\lambda_1(m) \lambda_2(n)}{(MN)^{1/2+s}} \bar{W}_1\left(\frac{m}{M}\right) \bar{W}_2\left(\frac{n}{N}\right) q^{2s} \frac{ds}{s}$$

where $\bar{W}_i(x) = W_i(x) x^{-1/2-\varepsilon-it}$ ($i=1, 2$) are compactly supported on $[1/2, 2]$ and satisfy $W_i^{(j)} = q^{j\varepsilon}$ for $j \geq 0$. $M = 2^\alpha$ and $N = 2^\beta$ satisfy $MN \leq q^{2+\varepsilon}$ (at the cost of negligible error).

By Stirling's formula, $\mathcal{V}(s)$ has exponential decay on the vertical line, so that we can truncate the integral at $\Im s \leq (\log 5q)^2$ at the negligible cost. Therefore, from equation (9) we have

$$\ll q^{1+\varepsilon} \sum_{d|q} \phi(d) \max_{M, N} \int_{-(\log 5q)^2}^{(\log 5q)^2} |\mathcal{V}(\varepsilon + it)| \\ \times \sum_{a, b, c, \delta \leq q^{\eta/200}} \frac{1}{\sqrt{abc\delta}} |ET(MN)| dt, \quad (10)$$

with

$$ET(MN) = \frac{1}{q\sqrt{MN}} \\ \times \sum_{\substack{ma\delta \equiv nbc(\bmod d) \\ ma\delta \neq nbc}} \lambda_1(m) \lambda_2(n) \bar{W}_1\left(\frac{m}{M}\right) \bar{W}_2\left(\frac{n}{N}\right), \quad (11)$$

Here we remove $(mn, q) = 1$ and the resulting error is $O(q^{-1/2+\eta/40+\varepsilon})$.

To ease notation, we define the exponents μ, ν by

$$M = q^\mu, N = q^\nu$$

By $1 \leq MN \leq q^{2+\varepsilon}$ we have $0 \leq \mu + \nu \leq 2 + \varepsilon$.

By symmetry, we may assume $0 \leq \mu \leq \nu \leq 2 + \varepsilon$ that and following [3, Section 6.2], we partition μ, ν into the following ranges for estimation.

(1) When $0 \leq \mu + \nu \leq 2 - \frac{3}{50}\eta$, for equation (11), we

have

$$ET(M, N) \ll \frac{(MN)^{\frac{1}{2}}}{q} q^{\frac{\eta}{100} + \varepsilon}$$

$$\ll q^{-1+\frac{\mu+\nu}{2}+\frac{\eta}{100}+\varepsilon}$$

$$\ll q^{-\frac{1}{50}\eta+\varepsilon}.$$

(2) When $2-\frac{3}{50}\eta \leq \mu+\nu \leq 2+\varepsilon, \nu-\mu \leq 1-\frac{7}{50}\eta$,

for equation (11), according to [3, Theorem 3.2], we have

$$ET(M, N) \ll q^{\frac{3\eta+\varepsilon}{200}} \left(\frac{N}{qM}\right)^{\frac{1}{4}} \left(1 + \left(\frac{N}{qM}\right)^{\frac{1}{4}}\right) + q^{-\frac{1}{2}+\frac{\eta}{40}+\varepsilon}$$

$$\ll q^{\frac{3\eta+\varepsilon}{200}} q^{-\frac{1}{4}+\frac{1}{4}(1-\frac{7}{50}\eta)} (1 + q^{-\frac{1}{4}+\frac{1}{4}(1-\frac{7}{50}\eta)})$$

$$\ll q^{-\frac{1}{50}\eta+\varepsilon}.$$

(3) When $2-\frac{3}{50}\eta \leq \mu+\nu \leq 2+\varepsilon, \mu+\nu^* \leq 1-\frac{1}{25}\eta$,

$N^* = q^2/N$ and $\nu^* = 2-\nu$, for equation (11), according to [3, (6.4)] and Lemma 5, we have

$$ET(M, N) = \frac{1}{\sqrt{MN}} \sum_{\substack{ma\bar{d}=nbc(\text{mod } q) \\ ma\bar{d} \neq nbc}} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{W}_2\left(\frac{n}{N}\right)$$

$$- \frac{1}{q\sqrt{MN}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{W}_2\left(\frac{n}{N}\right)$$

$$= \frac{1}{q\sqrt{MN}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{W}_2\left(\frac{n}{N}\right)$$

$$\times \left(1 + \sum_{r(\text{mod } q)}^* e\left(\frac{a\bar{d}mr - bcnr}{q}\right)\right)$$

$$- \frac{1}{q\sqrt{MN}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{W}_2\left(\frac{n}{N}\right)$$

$$= \frac{1}{q\sqrt{MN}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{W}_2\left(\frac{n}{N}\right) \sum_{r(\text{mod } q)}^* e\left(\frac{a\bar{d}mr - bcnr}{q}\right)$$

$$= \frac{1}{q\sqrt{MN}} \sum_{m \geq 1} \lambda_1(m)\bar{W}_1\left(\frac{m}{M}\right) \sum_{r(\text{mod } q)}^* e\left(\frac{a\bar{d}mr}{q}\right)$$

$$\times \sum_{n \geq 1} \lambda_2(n)\bar{W}_2\left(\frac{n}{N}\right) e\left(\frac{-bcnr}{q}\right)$$

$$= \frac{1}{q\sqrt{MN}} \sum_{m \geq 1} \lambda_1(m)\bar{W}_1\left(\frac{m}{M}\right) \sum_{r(\text{mod } q)}^* e\left(\frac{a\bar{d}mr}{q}\right)$$

$$\times \frac{N}{q} \sum_{n \geq 1} \lambda_2(n)\bar{W}_2\left(\frac{n}{q^2/N}\right) e\left(\frac{bcnr}{q}\right),$$

using Lemma 5, we transform the last term in the above expression. Given the property of \bar{W}_2 , n is bounded by $(q^2/N)q^\varepsilon$, where $N' \ll (q^2/N)q^\varepsilon$, (the resulting error term is negligible). By Lemma 5, it can be further simplified to

$$ET(M, N) = \frac{1}{q\sqrt{MN'}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{\omega}_2\left(\frac{n}{N'}\right)$$

$$\times \sum_{r(\text{mod } q)}^* e\left(\frac{a\bar{d}mr + \bar{b}c\bar{r}n}{q}\right)$$

$$= \frac{1}{q\sqrt{MN'}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{\omega}_2\left(\frac{n}{N'}\right)$$

$$\times S(a\bar{d}bc\bar{m}\bar{n}, 1, q)$$

$$\ll q^{-\frac{1}{2}+\varepsilon} (MN')^{\frac{1}{2}}$$

$$\ll q^{-\frac{1}{2}+\varepsilon} q^{\frac{\mu+\nu^*}{2}}$$

$$\ll q^{-\frac{1}{50}\eta+\varepsilon}.$$

(4) When $2-\frac{3}{50}\eta \leq \mu+\nu \leq 2+\varepsilon$ and in summary, we

have $1-\frac{1}{25}\eta \leq \mu+\nu^* \leq 1+\frac{7}{50}\eta$, for equation (11). From case (3), we have

$$ET(M, N) = \frac{1}{q\sqrt{MN'}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{\omega}_2\left(\frac{n}{N'}\right)$$

$$\times S(amb\bar{n}, 1, q)$$

$$= \frac{1}{q\sqrt{MN'}} \sum_{m,n} \lambda_1(m)\lambda_2(n)\bar{W}_1\left(\frac{m}{M}\right)\bar{\omega}_2\left(\frac{n}{N'}\right)$$

$$\times q^{\frac{1}{2}} Kl_2(amb\bar{n}; q)$$

$$= \frac{1}{\sqrt{qMN'}} \sum_{m \geq 1} \lambda_1(m)\bar{W}_1\left(\frac{m}{M}\right)$$

$$\times \sum_{n \geq 1} \lambda_2(n)\bar{\omega}_2\left(\frac{n}{N'}\right) Kl_2(amb\bar{n}; q),$$

applying Lemma 8, we have

$$ET(M, N) \ll \frac{q^\varepsilon}{\sqrt{qMN'}} \left(\sum_m |\lambda_1(m)|^2\right)^{\frac{1}{2}} \left(\sum_n |\lambda_2(n)|^2\right)^{\frac{1}{2}} (MN')^{\frac{1}{2}}$$

$$\ll \left(\frac{MN'}{q}\right)^{\frac{1}{2}} \left(M^{-\frac{1}{2}} + (MN')^{-\frac{3}{16}} q^{\frac{11}{64}}\right)$$

$$\ll q^{\frac{\mu+\nu^*-\varepsilon}{2}} \left(q^{\frac{\mu}{2}} + q^{-\frac{3}{16}(\mu+\nu^*)+\frac{11}{64}}\right)$$

$$\ll q^{-\frac{1}{4}+\frac{1}{20}\eta+\varepsilon} + q^{-\frac{1}{64}+\frac{7}{160}\eta},$$

taking $\eta = \frac{1}{5}$, the above expression satisfies

$$ET(M, N) \ll q^{-\frac{1}{50}\eta+\varepsilon}, \text{ in summary,}$$

$$ET(M, N) \ll q^{-\frac{1}{50}\eta+\varepsilon}, \quad (12)$$

therefore, combining (8) and (11), we have

$$E_1 \ll q^{-\frac{1}{50}\eta+\varepsilon}. \quad (13)$$

Combining the results from equations (7), (8), and (13), we obtain

$$S_1 \gg q(\log q)^{2k^2}.$$

By applying similar arguments to S_2 an equivalent lower bound can be obtained. According to equation (6), the proof of Proposition 1 is thus completed.

5. The Proof of Proposition 2

The proof of Proposition 2 is a special case of that in [18]. For brevity, we omit the details here. Therefore,

$$\sum_{\chi(\text{mod } q)}^* |\mathcal{N}(\chi, k-1)\mathcal{N}(\bar{\chi}, k)|^{2k-1} \ll q(\log q)^{2k^2}.$$

6. Conclusion

We Building on the work of Chen et al. and that of Blomer and others—who established an asymptotic formula with power-saving error terms for the first moment of the mixed product of character-twisted L -functions at the central point

when the modulus q of the Dirichlet character is prime-this paper investigates lower bounds for the mean value of the mixed product of two character-twisted L -functions at the central point under the same prime modulus condition. As a result, strong lower bound estimates are obtained for all continuous moments greater than one.

Acknowledgments

I would like to express my gratitude to my mentors for their support and guidance.

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