Study on Learning Difficulties of Function Concepts and Solutions

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Abstract: Mathematical concepts are the cornerstone of mathematics, the foundation of students’ cognition, and the carrier of students’ mathematical thinking activities. Mathematical concepts are learned in stages, and students may make errors at all stages, which in turn affect the quality of subsequent learning of related mathematical content. This paper analyzes and summarizes the difficulties in each stage of learning mathematical concepts from the student’s point of view, drawing on the “Action-Process-Object-Scheme” stage of learning mathematical concepts according to the APOS theory. Combined with the corresponding teaching strategies proposed by the APOS theory, we guide and help students to feel, obtain, understand and order to give some insights into the effective learning and teaching of mathematical concepts.

Keywords: Learning Difficulties, Concept Understanding, Concepts of Functions.

1. Concerns

Concepts are pivotal to teaching and learning, and the foundational and instrumental nature of mathematical concepts inclines math teachers to encourage students to deepen their understanding of concepts as they apply them. The teaching and learning process is often simplified and the widespread use of concepts seems to lead to high learning efficiency. But the fact is that math learning is often highly hidden, and knowing how to solve and manipulate does not necessarily lead to true understanding. The absence of teaching components affects the richness and comprehensiveness of students’ conceptual constructs. The American educator Dubinsky puts forward the APOS learning theory for mathematics, whose hierarchical view of conceptual construction provides a theoretical basis for teaching mathematical concepts layer by layer and is practicable in reality. What are the difficulties in studying conceptual learning in high school mathematics within the theoretical perspective of APOS? What are solutions for these learning difficulties? These have positive practical implications for promoting conceptual understanding of mathematics and improving learning and teaching effectiveness.

2. Overview of APOS Theory

Dubinsky and others proposed the APOS theory in the context of constructivism in the 1980s in response to the characteristics of mathematics learning. APOS is a combination of the first letters of “Action”, “Process”, “Object” and “Scheme”. According to this theory, the learning of mathematical concepts needs to go through four stages: action, process, object and scheme, among which, the “action stage” is for teachers to help students establish the relationship between intuitive backgrounds and abstract concepts through some external stimuli, and to feel the prototypes of mathematical concepts; “Process Stage” is to guide students to analyze and reflect on the concept of “Action Stage” repeatedly in their minds, to abstract the unique properties of the concept, to go through the process of internalization and compression, and to give it a formal definition and symbols; in “Object Stage”, students build on the unique properties of the concepts abstracted in the Process Stage to make further generalizations, ultimately resulting in refined and precise symbols, theorems, or definitions. In “Scheme Stage”, students reintegrate existing schematic structures with the special cases, abstraction processes, definitions and symbols of newly learned concepts to form a new comprehensive scheme.[1] APOS theory is a constructivist learning theory that emphasizes the process of concept formation and advocates that students construct cognitive structures and acquire knowledge and abilities through mathematical activities, which is in line with the six core literacies, the “four fundamentals” and the “four competencies” put forward by the new curriculum standards.

3. Learning Difficulties of High School Function Concepts under APOS Theory

Students’ learning difficulties begin with the definition of concepts.[2] Most middle school students do not know what a function is, and students find it easier to give examples than definitions.[3] However, some scholars have found that the examples used are often incorrect.[4] On the other hand, students have difficulties with language skills, logic skills, and abstraction skills when learning concepts.[5] Students have difficulty defining a function without using abstract mathematical terms and without proper vocabulary.[6] Students’ conceptual learning is characterized by a lack of thorough or biased understanding of concepts and a failure to grasp the essence of conceptual connotations and extensions.[7] Due to the influence of the types of functions that students use in their daily lives, they believe that there must always be a rule (an algebraic expression) to define a function. They even incorrectly assume that the rule is actually the function. [8] Students have difficulty distinguishing between equations and functions. [9] Finally, there is the phenomenon that students understand what they are taught in class, but when they do the problems on their own, they often fail to get the ideas, find the methods, and
don't know how to start, which is the phenomenon of “understand but can't do it”.[10] Problems in students’ conceptual learning of functions include a vague understanding of the essence of the concept of function, a vague perception between the connotation and extension of functions, and unfamiliarity with the application of functions.

[11] In summary, drawing on the APOS theory of “Action-Process-Object-Scheme” concept of learning stages, the learning difficulties and causes at each stage of function learning are summarized as follows: (1) Learning difficulties at action stage: bias in perceiving the conceptual prototype of a function due to the influence of the amount and typicality of visual background material; (2) Learning difficulties at process stage: the inability to abstract the common features of multiple functions and to summarize the essential features of functions due to weak generalization and abstraction skills, negative transfer of concepts related to daily life, and the influence of conceptual prototypes; (3) Learning difficulties at object stage: They believe that only the analytic formula can represent the correspondence of a function due to the confusion of the connotation and extension of the concept, the wrong understanding and application of mathematical concepts and related symbols, and the insufficient ability to convert the language of mathematical concepts. Their understanding of functions is generally limited to concrete, visual, typical and rich examples, combined with the visual background material; (2) Learning difficulties at action stage repeatedly, and then through the continuous process stage, in the continuous experience and comprehension of the experience and concepts, intuition and logic of the overall integration and cohesion, and cultivate the students' ability to abstract and generalize, sublimation of the formation of literacy.[13] Teachers need to pay attention to the concretization of mathematical concepts to understand things through perception and manipulation of the world. In teaching, it is important to relate concepts to specific contexts, and concretizing graphic demonstrations can promote students’ knowledge of related knowledge.

### 4.3. Solutions for Learning Difficulties in Object Stage

The object stage is to consider the “process” carried out in the previous process stage as a concrete and independent object, to recognize its essence and some other properties, and to summarize and sublimate it, so that it will eventually produce a refined and accurate symbol, theorem or definition. In the object stage, teachers are required to help students reveal the connotation of concepts, clarify the extension of concepts, strengthen the identification and generalization, analyze the meaning of the concepts and symbols, strengthen the mathematical concepts of the language of conversion training, learning mathematical concepts with classification, comparison, counter-examples and other ways to reveal the inner connection between concepts, to help students master a certain range of a large set of concepts.[14] Focusing on multiple representations of concepts provides students with multiple ways of understanding concepts and helps them to understand their richness in a more comprehensive manner.

### 4.4. Solutions for Learning Difficulties in Scheme Stage

In scheme stage, after the previous three stages are truly completed, the existing scheme structure is reintegrated with the newly learned knowledge to form a new synthesized scheme. At this stage, teachers need to lead students to strengthen concept analysis, clarify the difference between old and new concepts, neighboring concepts, establish the connection between concepts, and pay attention to the application of concepts. A scheme is a unit of knowledge, a block of knowledge, and a system of knowledge in the human brain that includes the relationship between the core concepts and the knowledge of how and when to apply the core concepts. Strengthening the construction and automation of schemes is conducive to the formation of good cognitive structures. Concept mapping pedagogy and the creation of concept maps can effectively contribute to teaching and learning and improve the quality of teaching and learning.[15] Classroom summaries focus on enabling students to construct mathematical concept maps and form a well-organized network of cognitive structures. Mind maps, circle maps, bubble maps, bracket maps, etc., can be fully utilized in the process of mathematics teaching and learning, so that students can experience the correlation of knowledge and the characteristics and advantages of each other in a multiform manner.
5. A Case of Application of Solutions for Learning Function Concepts under APOS Theory

5.1. Action Stage

To address the problem of bias in students’ perception of the function concept prototype, the teaching process is presented with rich and typical background materials to help students perceive the connection between the mathematical background and the function concept prototype. The correspondence between Materials 1 and 2 is the same, but the domain of definition and the domain of value are different, which leads students to pay more attention to the two key elements of the domain of definition and the domain of value of functions. The correspondences of the functions in Materials 3 and 4 are presented in the form of graphs and tables, leading students to realize that besides analytical formulas, graphs and tables are also representations of the correspondences of functions, and also allowing students to understand the correspondences contained in the graphs, which provide a basis for determining if they are functions or not.

<table>
<thead>
<tr>
<th>Seq.</th>
<th>Specific material</th>
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<tbody>
<tr>
<td>1</td>
<td>Material 1: Assuming that a “Fuxing” train runs at a constant speed for half an hour, the relationship between its traveled distance $S$ (unit: km) and running time $t$ (unit: h) can be expressed as $S = 350t$.</td>
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<tr>
<td>2</td>
<td>Material 2: Assuming that workers in a company work at least 1 day per week and at most 6 days per week, and that the company's wage rate is 350 yuan per person per day, paid once a week, the relationship between the wage $w$ and the number of days $d$ worked in a week $d$ is $w = 350d$.</td>
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<tr>
<td>3</td>
<td>Material 3: The figure below shows the change of Air Quality Index (AQI) in Beijing on November 23, 2016.</td>
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<tr>
<td>4</td>
<td>Material 4: Engel coefficient is one of the indexes reflecting the living standard of the people, the following table shows the data on the change of Engel coefficient of urban residents in a province of China.</td>
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These four materials are set up to guide students to experience the process of mathematical abstraction, identify mathematical properties in real-life situations, discard non-mathematical features and attributes, and form a “model” for further study. Under the step-by-step guidance of the teacher, help students to deeply participate in the activities, think carefully and explore bravely. Through the activities, the correct perception of the conceptual prototype of function is accomplished.

<table>
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<tr>
<th>Material</th>
<th>Thinking on Questions</th>
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| 1        | Question 1: According to the function concepts learned in junior high school, please determine whether $S$ is a function of $t$? Why?  
Question 2: Is it correct to say that “according to the correspondence $S = 350t$, this train will advance 350 km in 1 h”? How can the correspondence between $S$ and $t$ be expressed more precisely? | Allow students to engage in cognitive conflict as they solve problems and notice the range of variables and begin to describe the one-to-one correspondence in the range of variables. |
| 2        | Question 3: Following the example of Question 1, describe in precise terms the correspondence between $w$ and $d$ in Material 2.  
Question 4: The functions in Material 1 and Material 2 have the same correspondence, do you think they are the same function? Why? | Making students pay more attention to the domain of definition and the domain of value and appreciate their necessity for the concept of function. |
| 3        | Question 5: Do you think that $I$ is a function of $t$? If so, describe their correspondence following the previous method.  
Question 6: From the scheme, can we accurately describe the range of values of $I$? Why? | Making students realize that a scheme is also an expression of a function, and that the domain of value is a subset of the set in which the function's values are found, and having students point out the correspondences contained in the scheme provides a basis for determining whether or not it is a function. |
| 4        | Question 7: Do you think the Engel coefficient $r$ is a function of $y$?  
Question 8: Describe precisely their correspondence following the previous questions.  
Question 9: How has the quality of life of the residents of this provincial city changed? | Making students realize that a table is also a functional expression. |
5.2. Process Stage

In response to the difficulty of abstracting the common features of conceptual prototypes at the process stage, students are guided to become more involved in the process of abstracting the common features on the basis of four typical materials and nine heuristic questions during the action stage. First, students are guided to summarize the key elements of the function concept from each of the materials, all of which are generalized and summarized in terms of sets and correspondences; secondly, respecting students’ thinking and ideas, allowing students to fully demonstrate their own ideas, the teacher actively responds to the students while they express their own ideas, and helps them to answer the questions. Finally, the functions in the four materials are then represented according to the key elements of the function concept and further analyzed to obtain the common features of the function concept prototype. At that stage, the teacher also endeavors to guide the students to scaffold between their prior cognitive foundation and the new knowledge so that the students can reach the goal in time to accept the concept of function.

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<tr>
<th>Question 10: What are the common features of the functions in Materials 1 - 4 above? From this can you generalize the essential features of the concept of function?</th>
<th>Facilitate the generalization of essential features and guide students to find the direction of generalizing the common features of functions. On this basis, allow students to generalize that functions have the following common features: ① both have two non-empty number sets, denoted by A and B; ② both have a correspondence.</th>
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</table>
| Question 11: How are independent and dependent variables linked through correspondence? | Promote generalization of essential characteristics. Teachers guide students to get: Although expressed differently, all four functions satisfy the requirement that “for every x in the set A, there is a uniquely determined value of y in the set B that corresponds to it according to some correspondence”.

The end of the action stage also allows students to learn a new mathematical idea: from the particular to the general. The penetration of this idea helps students to deepen their memory of function concepts; it helps students to form function concept prototypes in their minds. At this point, the teacher emphasizes the symbolic meaning and considerations of function concepts, which can help students to form a prototype of function concepts, as well as pave the way for the next stage of forming the object.

5.3. Object Stage

In object stage, students believe that only the analytic formula can represent the correspondence of a function, and their understanding of functions is generally limited to mathematical objects, ignoring the fact that a function is also a dynamic process of association or covariation. They have difficulties in distinguishing the one-to-one correspondence between the domain of definition and the domain of value, and they have difficulties in determining whether two functions are equal or not. At this stage, we pose questions on the essence of the function and some other properties of the key points, combined with positive and negative examples, to guide students to explore and think, and constantly generalize and sublimate, to reveal the essential characteristics of the concept, and ultimately to make it produce refined and accurate symbols, theorems or definitions, to form the final concept of the function.

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<tr>
<th>Question 12: What forms do correspondences in functions take?</th>
<th>Help students to construct the precise concept of function on the basis of the identification of key points and error-prone points, implement the generation of concepts, and realize the improvement of students’ generalization and summarization and expression and communication skills.</th>
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<tr>
<td>Question 13: How do the domain of definition and domain of value correspond in the concept of function?</td>
<td>Implement the application of the concepts by combining the forward application of finding the domain of definition and value of a function with the inverse application of using counterexamples to determine which is a function and which two functions are the same. Using the concept of function repeatedly, students will be able to solve function concept problems with the “function concept” as a whole.</td>
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| Question 14: Find the domain of definition and the domain of value of the following function. 
\[ f(x) = \frac{1}{4x^7}, \quad f(x) = \sqrt[1-x^3]{x} \] |  |
| Question 15: Which of the following are functions? 
\[ f(x) = 1, \quad y = x, x \in [0, 6], y \in [0, 3] \] |  |
| Question 16: Which of the following pairs of functions are equal? 
The function \( h = 130t - 5t^2 \) that expresses the relationship between the height h and the time t of the flight of the projectile and the quadratic function \( y = 130x - 5x^2 \); \[ f(x) = 1 \text{ and } g(x) = x^5 \]. |  |
5.4. Scheme Stage
At the generalization stage, there are difficulties in establishing connections between old and new knowledge and in distinguishing between equations and functions. At this stage, students will be guided to analyze the three elements of the functions they have already learned in relation to the newly learned concepts of functions to establish the connection between the old and the new knowledge. They will also be guided to further clarify the relationship between functions and their corresponding equations by drawing graphs.

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<td>Question 17: Analyze a function you have already studied based on its definition: What are the domains of definition, domains of value, and correspondences of primary, inverse proportional, and quadratic functions? Compare the similarities and differences between the functions you have already studied and the functions you are studying now.</td>
<td>Starting with functions that students are familiar with, students can use the concepts of the functions they are learning to explain the functions they have already learned, thus strengthening their understanding of function concepts. Deepen students' understanding of presently learned concepts by comparing the similarities and differences between the function concepts they have already learned and the present function concepts.</td>
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<td>Question 18: What are the differences and connections between primary functions and primary equations, inverse proportional functions and fractional equations, and quadratic functions and quadratic equations?</td>
<td>Lead students to strengthen conceptual analysis, clarify the difference between old and new concepts and neighboring concepts, establish connections between concepts, and strengthen the construction of schemes.</td>
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References