International Journal of Education and Humanities
ISSN: 2770-6702 | Vol. 12, No. 2, 2024

Summary and Reflection on the Teaching of Functions in Calculus for Economics and Management

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Abstract: This article takes the research object of calculus - functions as the starting point, sorts out the manifestation of composite functions in various operation stages of calculus, summarizes the diversity of function forms, proposes teaching thoughts and suggestions for some key and difficult contents, and demonstrates the specific application of functions in calculus through examples.

Keywords: Function; Calculus; Variable Replacement.

1. Introduction

Compared to the Mathematical Analysis course in professional mathematics, the Calculus course offered in many economics and management majors has a slightly lower difficulty level, and the basic theoretical content is relatively simple, with a greater emphasis on application. Due to various factors such as differences in humanities and sciences among students, regional differences, changes in teaching and learning methods, and some content being disconnected from middle school mathematics, Calculus is a challenging course for many freshmen in the field of economics and management. In order to help students better and faster grasp the content of calculus, this article takes the research object of calculus - functions as the starting point, summarizes the diverse changes and key and difficult applications of functions in calculus, in order to provide reference for calculus teaching and the learning of college freshmen.

The main research content of calculus is to perform operations on functions such as limits, derivatives, and integrals. Ruofeng Zhang [1] summarized that the calculus part of a single variable function mainly involves six concepts, namely limit, continuity, derivative, differential, indefinite integral, definite integral, as well as three theorems: differential mean value theorem, integral mean value theorem, and basic theorem of calculus (Newton Leibniz formula). Xinying Pai [2] summarize the calculus of unary functions as "one tool - limit; two operations - differentiation and integration; three concepts - continuity, derivative, and integration; four calculations - limit, derivative, integration, and differential equations; five types of functions - elementary functions, piecewise functions, parametric equations, implicit functions, and integral upper limit functions". Among them, functions are the foundation and core of calculus, and the main research object of calculus. They describe the interdependence between variables. From the perspective of the number of independent variables, it can be divided into univariate functions and multivariate functions, with univariate functions as the foundation. This article focuses on discussing univariate functions.

2. A Review of The Concept of Functions

The concept of functions has been learned since middle school, and whether it is in high school textbooks or university textbooks, the concept of functions is unified (Definition 1 [3]).Further expand from Definition 1 to the definitions of inverse and composite functions (Definition 2 [3] and Definition 3 [3]).

Definition 1: Let x and y be two variables, and D be a nonempty set of numbers. If according to a certain correspondence rule f , for each $x \in D$, the variable y has a unique and definite value corresponding to it, then this correspondence rule f is called a function defined on D , commonly represented by $y = f(x)$.

Definition 2: Let the domain of the function $y = f(x)$ be D and the range of values be W . If there is a unique $x \in D$ for any real number y in W , let $f(x) = y$. So let's consider y as the independent variable and x as the causal variable. Based on the definition of the function, x becomes a function of y , and this function is called the inverse function of $y = f(x)$, denoted as $x = f^{-1}(y)$. Its domain of definition is W , and its range of values is D .

Definition 3: Let the function $y = f(u)$ be defined in the domain D_f .The definition domain of function $u = g(x)$ is D_g , and the value domain is W_g . If $W_g \cap D_f \neq \Phi$ is met, then the function $y = f[g(x)]$ $x \in \{x \mid g(x) \in D_f\}$ is called a composite function composed of functions $y = f(u)$ and $u = g(x)$, where u is referred to as the intermediate variable.

Usually, six types of functions, including constant value function, power function, exponential function, logarithmic function, trigonometric function, and inverse trigonometric function, are collectively referred to as basic elementary functions. A function obtained from a basic elementary function through finite number of arithmetic operations and finite number of composite operations, which can be expressed in one formula, is called an elementary function.

3. Specific Applications of Functions in Calculus

3.1. Application of Composite Function Concept in Calculus

The limits, derivatives, differentiation, integration, etc. of basic elementary functions are the basic contents of calculus and are relatively simple. The composition of functions makes the forms of functions more diverse and flexible, mainly reflected in the application of variable replacement ideas and methods. The theory of composite functions has corresponding knowledge systems in the processes of finding limits, derivatives, differentiation, and integration. The summary is summarized as follows:

(1) Variable Replacement Theorem in Extreme Operations

Theorem 1: Let $y=f(u)$ and $u=g(x)$ form a composite function $y=f[g(x)]$. If $\lim_{u\to u_0} f(u)=A$, $\lim_{x\to x_0} g(x)=u_0$, and $g(x)\neq u_0 \, (x\neq x_0)$, then there is a conclusion $\lim_{x\to x_0} f[g(x)]=\lim_{u\to u_0} f(u)=A$.

The application of the variable replacement theorem makes the form of the function for finding limits more diverse, such as $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\sin x^2}{x^2}=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=1$.

(2) Chain rule for taking derivatives of composite functions in derivative operations

Theorem 2: If the function $u=\varphi(x)$ is differentiable at point x and the function $y=f(u)$ is differentiable at point $u=\varphi(x)$, then the composite function $y=f[\varphi(x)]$ is differentiable at point x and has formula $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$ or $y'=f'(u)\cdot\varphi'(x)$.

When solving the derivative of a composite function, according to the chain rule, first decompose which basic elementary functions the given function is composed of, and then calculate the product of the derivative of each function layer with respect to its independent variables in order from the outside to the inside.

The differentiation rule of composite functions is based on the differentiation formula of basic elementary functions. The differentiation problem of elementary functions can generally be solved by the composite function differentiation rule and the derivative four operation rule.

(3) Differential rules for composite functions in differential operations

Theorem 3: If both $y=f(u)$ and $u=\varphi(x)$ are differentiable, then the differentiation of the composite function $y=f[\varphi(x)]$ is $dy=\{f[\varphi(x)]\}'dx=f'(u)\varphi'(x)dx=f'(u)du$.

Obviously, regardless of whether u is an independent variable or an intermediate variable, the differentiation of a function always maintains the same form, that is to say, the differential form is invariant.

(4) Integral substitution method in integral operations

Taking indefinite integrals as an example, there are two types of substitution integration methods based on known conditions:

Theorem 4 (First Substitution Integration Method): If $\int f(x)dx=F(x)+C$, then

$$\int f[\varphi(x)]\varphi'(x)dx=\int f[\varphi(x)]d\varphi(x)=F[\varphi(x)]+C,$$

where $\varphi(x)$ has a continuous derivative.

Theorem 5 (Second Substitution Integration Method): Let $x=\varphi(t)$ be monotonic, differentiable, and $\varphi'(t)\neq 0$, its inverse function is $t=\varphi^{-1}(x)$. If $\int f[\varphi(t)]\varphi'(t)dt=F(t)+C$, then $\int f(x)dx=\int f[\varphi(t)]\varphi'(t)dt=F(t)+C$ $=F[\varphi^{-1}(x)]+C$.

By substituting appropriate variables, a complex integration can be transformed into a simple integration or a known integration formula. For example, the first substitution method can be used to calculate the integral $\int x^2\sin(x^3+1)dx$ $=\frac{1}{3}\int\sin(x^3+1)d(x^3+1)$ $=-\frac{1}{3}\cos(x^3+1)+C$

3.2. Diversification and Application of Function Forms

Compared to middle school mathematics, calculus has a more diverse range of function types, such as power exponential function $y=u(x)^{v(x)}$ $(u(x)>0)$, which is a type of elementary function that can be transformed into $u(x)^{v(x)}=e^{v(x)\ln u(x)}$ using formulas. Power exponential function often appear in operations such as finding limits and derivatives. In addition to elementary functions, there are also various forms such as piecewise functions, functions determined by parametric equations, implicit functions, integral upper limit functions, and integrals with parametric variables.

(1) Piecewise function

The characteristic of piecewise functions is that they are represented by different expression in different parts of their domain of definition. It is particularly emphasized that a piecewise function is a function in its domain of definition, rather than multiple functions. Common piecewise functions include absolute value functions $y=|x|$, sign functions $y=\operatorname{sgn} x$, rounding functions $y=[x]$, etc. The study of continuity, differentiability, and integrability of piecewise functions is a key focus of calculus, especially the discussion of limits, continuity, and differentiability at piecewise points.

(2) Functions determined by parametric equations

The function $y=f(x)$ of y and x represented by the parametric equation $\begin{cases}x=x(t)\\ y=y(t)\end{cases}$ is called the function determined by the parametric equation. Finding the derivative of a function determined by a parametric equation is the focus of examination. If the parameters are easily eliminated, the analytical formula $y=f(x)$ can be obtained first, and then the derivative can be taken; If it is difficult to eliminate parameters or you do not want to use the parameter elimination method, you can use the composite function differentiation and inverse function differentiation methods to obtain $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}/\frac{dx}{dt}=\frac{y_t'}{x_t'}$ .Similarly, the second derivative can be obtained as follows, $\frac{d^2y}{dx^2}=\frac{d}{dt}(\frac{dy}{dx})\cdot\frac{dt}{dx}$ $=\frac{d}{dt}\left(\frac{y_t'}{x_t'}\right)\cdot\frac{1}{x_t'}=\frac{x_t'y_t''-y_t'x_t''}{(x_t')^3}$. By understanding the bridging effect of parameter t and following this process, higher order derivatives can be calculated.

(3) Implicit function

The functional relationship between variables x and y is represented by equation $F(x,y)=0$, which is called the implicit function determined by the equation. Some implicit functions can be transformed into explicit functions, which can also be handled using the method of explicit functions.

The theoretical part of implicit functions in calculus textbooks for economics and management is briefly described, generally assumed to be discussed under the assumption that there is no misunderstanding and the uniqueness of implicit functions is met. But students may have some questions during their learning, such as "Can the equation $F(x,y)=0$ always determine the implicit function? If so, is the only $y=f(x)$ determined?" So teachers need to explain the relevant theories clearly during the teaching process.

Not every equation can determine the implicit function, such as $x^2+y^2+c=0(c>0)$ .For example, $x^2+y^2=1$ can determine $y=\sqrt{1-x^2}$ defined on $[-1,1]$ or $y=-\sqrt{1-x^2}$ defined on $[-1,1]$. In this case, the range of values for x and y must be given to make sense. The uniqueness theorem and differentiability theorem of implicit functions ensure the rigor of implicit function operations [4].

The application of implicit functions mainly focuses on derivatives, which is the main content of implicit functions in calculus textbooks. The commonly used method is to use the chain rule, treat y as an intermediate variable, take direct derivatives on both sides of the equation, and solve for y' , or use differential methods that are invariant in differential form, which will not be illustrated here. The integration operation of implicit functions is slightly more difficult than the differentiation operation, and is often solved by introducing parameter methods and transforming them into functions determined by parameter equations. However, parameterization techniques are more flexible, and commonly used methods include linear substitution or ratio substitution, as shown in the following examples.

Example 1: Let $y=y(x)$ be the implicit function determined by equation $y(x-y)^2=x$, calculate $\int\frac{1}{x-3y}dx$.

Solution: Let $x-y=t$, substitute it into equation, and solve for $x=\frac{t^3}{t^2-1}$, $y=\frac{t}{t^2-1}$, then $dx=\frac{t^2(t^2-3)}{(t^2-1)^2}dt$,so

$$\int\frac{1}{x-3y}dx=\int\frac{1}{\frac{t^3}{t^2-1}-\frac{3t}{t^2-1}}\cdot\frac{t^2(t^2-3)}{(t^2-1)^2}dt=\int\frac{t}{t^2-1}dt$$

$$=\frac{1}{2}\ln\left|t^2-1\right|+C=\frac{1}{2}\ln\left|(x-y)^2-1\right|+C$$

Parameters can also be introduced in the form of $y=tx$ or $x=ty$, and readers can calculate them themselves.

(4) Integral upper limit function

If the function $f(x)$ is continuous on $[a,b]$, then for any $x\in[a,b]$, $f(x)$ is continuous on $[a,x]$, and therefore can also be integrated. Therefore, $\int_a^x f(t)dt$ exists as a function of the upper limit x , which is called an integral upper limit function or a variable upper limit integral. $\int_a^x f(t)dt$ is an original function of $f(x)$ on $[a,b]$, that is, $[\int_a^x f(t)dt]'=f(x)$, $x\in[a,b]$, which is called the existence theorem of the original function. With the integral upper limit function and the existence theorem of the original function, it is easy to derive the Newtonian Leibniz formula $\int_a^b f(x)dx$ $=F(x)\Big|_a^b=F(b)-F(a)$ for calculating definite integrals, where $F'(x)=f(x)$. The chain rule for taking derivatives of composite functions has the formula $[\int_a^{\varphi(x)} f(t)dt]'$ $=f[\varphi(x)]\varphi'(x)$, where $\varphi(x)$ is a differentiable function.

The integral upper limit function extends a new form of function and has applications in many knowledge points of calculus, especially in the type of differentiation using the integral upper limit function. The common question types are as follows:

Calculation of undetermined type limits with variable limit integrals, such as " $\lim\limits_{x\to1}\frac{\int_x^1(t^2-1)dt}{x-1}=\lim\limits_{x\to1}\frac{[\int_x^1(t^2-1)dt]'}{(x-1)'}$ $=\lim\limits_{x\to1}(1-x^2)=0$ ", which uses the Roberta rule to determine the limit.

The solution of equations with variable limit integrals, such as "let the function $f(x)$ be continuous and meet $\int_0^x f(x-t)dt=\int_0^x(x-t)f(t)dt+e^{-x}-1$, calculate $f(x)$." This question is for the 2016 Mathematics Third Graduate Entrance Examination. On the left side of the equation, first replace the elements, let $x-t=u$, in this process x is treated as a constant, and when taking the derivative on both sides of the equation, x is treated as a variable.

Construct an uncertain limit integral function to prove the conclusion, such as "let the functions $f(x)$ and $g(x)$ be continuous on $[a,b]$, prove the existence of at least one point $\xi\in(a,b)$, so that $f(\xi)\cdot\int_\xi^b g(x)dx=g(\xi)\int_a^\xi f(x)dx$ ". An auxiliary function $F(x)=(\int_a^x f(t)dt)\cdot(\int_x^b g(t)dt)$ can be constructed, and the proof can be obtained by using the Rolle's mean value theorem.

There are also monotonicity criteria for integral upper limit functions, extreme value calculations, and so on. It can be seen that integral upper limit functions are a very important class of functions in calculus, with wide applications.

(5) Integral with parametric variables

Integral with parameters generally includes normal integral with parameters, generalized integral with parameters, etc. The Gamma function $\Gamma(s)=\int_0^{+\infty} x^{s-1}e^{-x}dx$ involved in calculus converges when $s>0$, which is one of the generalized integrals with parameters. There are basic properties $\Gamma(s+1)=s\Gamma(s)\ (s>0)$ and $\Gamma(n+1)=n!$, indicating that Gamma functions are an extension of factorial functions. The use of Gamma functions and recursive formulas can make certain integration operations more concise and avoid multiple partial integration operations. Such as $\int_0^{+\infty} x^7 e^{-x^2}dx \underline{\underline{x^2=t}} \frac{1}{2}\int_0^{+\infty} t^3 e^{-t}dt$ $=\frac{1}{2}\Gamma(4)=\frac{1}{2}\cdot 3!=3$ .Gamma functions are particularly widely used in subsequent courses such as Probability Theory and Mathematical Statistics.

Sometimes there may be a mixture of several types of functions, such as implicit functions and functions determined by parameter equations in the following example.

Example 2: Let $y=y(x)$ be a differentiable function determined by equation system $\begin{cases} x=2t-1 \\ te^y+y+1=0 \end{cases}$, calculate $\frac{dy}{dx}\big|_{t=0}$.

Solution: Take the derivative of the first equation to obtain $x'_t=2$, take the derivative of the second implicit function equation to obtain $e^y+te^y y'_t+y'_t=0$, and solve for $y'_t=-\frac{e^y}{te^y+1}$. Using $\frac{dy}{dx}=\frac{dy}{dt}/\frac{dx}{dt}=\frac{y'_t}{x'_t}$, we obtain $\frac{dy}{dx}=\frac{-e^y}{2(te^y+1)}$ .Substituting $t=0$ into the equation yields $x=-1$, $y=-1$, so $\frac{dy}{dx}\Big|_{t=0}=-\frac{1}{2e}$.

A similar question is as follows: "let the differentiable function $y=y(x)$ be determined by equation $\int_0^{x+y} e^{-t^2}dt=\int_0^x x\sin^2 t dt$, calculate $\frac{dy}{dx}\big|_{t=0}$." There is a mixture of implicit function and integral upper limit function in the equation.

3.3. Application of Auxiliary Function Construction Method

If a function is not directly given in the problem and a function analytical expression is needed to solve the problem, auxiliary functions can be constructed according to the needs of the problem, such as using zero point theorem, differential mean value theorem to prove the equality, and using function monotonicity, Lagrange mean value theorem, maximum theory to prove inequalities, clear function analytical expressions are required. There are various methods for constructing auxiliary functions, which need to be comprehensively considered based on the specific structure to be proven. The construction of auxiliary functions in proving problems using the differential mean value theorem can be directly constructed based on observation methods, such as the construction of variable limit integral functions mentioned in the application of integral upper limit functions in the previous text. Some complex forms can be comprehensively refined by combining integral results and differential equation solving methods. For example, to prove a form such as $f'(\xi)+g(\xi)f(\xi)=0$, we can construct the function $F(x)=e^{G(x)}f(x)$, where $G'(x)=g(x)$, and apply the Rolle's mean value theorem. As the differential equation $f'(x)+g(x)f(x)=0$ is a separable variable equation, its general solution is $f(x)=Ce^{-\int g(x)dx}$, which holds for $e^{\int g(x)dx}f(x)=C$, where C is an arbitrary constant, an auxiliary function $F(x)=e^{G(x)}f(x)$ can be constructed.

Example 3: Let $f(x)$ be second-order differentiable on $[0,1]$, and $f(0)=f(1)=0$, prove the existence of $\xi\in(0,1)$, such that $f''(\xi)=\frac{2f'(\xi)}{1-\xi}$.

Proof: The structure to be proven is $f''(\xi)+\frac{2f'(\xi)}{\xi-1}=0$, and because of $[2\ln(x-1)]'=\frac{2}{x-1}$, $F(x)=(x-1)^2 f'(x)$ can be set, obviously $F(1)=0$. $f(x)$ meets the Rolle's mean value theorem on $[0,1]$, therefore there exists $\xi_1\in(0,1)$, which holds $f'(\xi_1)=0$, and $F(x)$ also meets the Rolle's mean value theorem on $[\xi_1,1]$, therefore there exists $\xi\in(\xi_1,1)\subset(0,1)$, which holds $F'(\xi)=0$. $F'(\xi)$ $=2(\xi-1)f'(\xi)+(\xi-1)^2 f''(\xi)=0$, namely $f''(\xi)=\frac{2f'(\xi)}{1-\xi}$.

The method of constructing auxiliary functions is relatively flexible and also a difficult point in the teaching process. For various question types, students should be guided to learn how to summarize and start from the problem, using appropriate functions and related theoretical knowledge to solve the problem.

4. Conclusion

Functions are the main research object of calculus operations, and no matter how diverse their forms are, their essence is to reflect the dependency relationships between variables. When applying, it is necessary to distinguish the roles of each variable, master some commonly used variable substitution methods and techniques, be familiar with the main applications of some special types of functions, and be able to extend from univariate functions to multivariate calculus. Combining the main operations of calculus such as limits, differentiation, and integration, make good use of this subject object. In addition, simple economic functions such as supply and demand functions, cost functions, revenue functions, and profit functions are also introduced in the calculus of economics and management. These functions are often used in marginal analysis, elasticity analysis, and extreme value calculation, reflecting the widespread application of calculus as a tool in practical economic problems.

Acknowledgment

This study was supported by Fund Projects: Anhui

Provincial Higher Education Teaching and Research Project (2022jyxm016).

References

[1] R. F. Zhang. Application of Variational Limit Functions in Calculus. Journal of Tianshui Normal University, 2002 (05): 57-58.

[2] X. X. Pai, Y. M. Hou, J. Yu. "One, Two, Three, Four, and Five" in Calculus of Univariate Functions. Advanced Mathematics Research, 2022, 25 (06): 4-6.

[3] H. Q. Yang. Economic Mathematics Calculus (3rd Edition) (Micro Course Edition). Beijing: People's Posts and Telecommunications Press, 2020.

[4] Mathematics Department of East China Normal University. Mathematical Analysis (Second Edition). Beijing: Higher Education Press, 1991.