

# Super-stable Kneading Sequences with Double Cycles in 1D Bimodal Maps

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**Abstract:** It is well known that a super-stable kneading sequence (SSKS) is an important concept, all SSKSs in bimodal maps forms joints in the corresponding symbolic dynamics, it decides the multiplication table of star products, which the n-tupling bifurcations to chaos can be investigated and Feigenbaum's metric universalities can be measured and reconstructed, this SSKSs have form which are periodic with single cycle. However, in fact, the SSKSs in bimodal maps have another form with double cycles which are little mentioned and researched, they have the same position and significance as the single cycle SSKS. In the paper, we presented the number of admissible SSKSs with period-n and the joints graph on the parameter plane.

**Keywords:** Symbolic dynamics; SSKS with double cycles; Bimodal maps.

## 1. Introduction

Symbolic dynamics in 1D bimodal maps have attained many achievements. For example, MacKay proposed that the "flesh", "skeleton" and "joints" constructed bifurcation structure in the symbolic space[1], the period doubling routes to chaos for bimodal maps was presented right after [2-3]; Renormalization in bimodal maps was generalized by Veitch [4], Ringland and Milnor presented a genealogy for finite kneading sequences of bimodal maps and entropy and monotonicity for real cubic maps respectively [4-5], Base on the work above and equal-entropy algorithm, Peng presented the dual star products based on the SSKS with single cycle [6]. All this dedication for bimodal maps enriched the foundation contents of symbolic dynamics and accelerated the research of symbolic dynamics with multi-modal maps.

However, all the work were based on an important concept, the so called SSKS, it was defined in the Hao B.L. Book [7] as: A SSKS is a kind of periodic sequence which pass through all critical points in the iterated map, so a SSKS in uni-modal maps has the form  $(XC)^\infty$ , a SSKS in bimodal maps is  $(XDYC)^\infty$ , this kind of single-cycle SSKS is naturally researched and played a crucial role in fact. With the deep understanding of SSKS, we give an empirical and equivalent definition: Periodic sequences who can be calculated the parameters of the iterated map by the word lifting technique are called SSKS. In another word, A SSKS is a point of n-dimensional parameter space, the so-called "joints" in the symbolic space. The Newton iterative method is used to calculate the parameters of admissible SSKSs with single and double cycles. Later, in this paper, we presented some schematic iterative graphs of two SSKS as examples, scatter plots on parameter plane are given finally.

## 2. The Concise Description of Symbolic Dynamics in Bimodal Maps

### 2.1. SSKSs with Single Cycle and Double Cycles

#### 2.1.1. The General Bimodal Maps and Symbolic Dynamics

The 1D bimodal maps are cubic maps with parameters as follows,

$$f(x, \mu) = k \int (x-c)(x-d)dx + i \quad (1)$$

Consider endomorphism interval  $[-1,1]$  and two boundary conditions  $f(-1) = -1$  and  $f(1) = 1$ ,

$k$  and  $i$  are eliminated by  $k = \frac{3}{1+3*c*d}$  and  $i = \frac{k*(c+d)}{2}$ , the iterative equation is rewritten as (2), here,  $c$  and  $d$  are horizontal coordinates of two critical points  $C$  and  $D$  respectively,  $\mu = (c, d)$ .

$$x_{n+1} = f(x_n, \mu) \quad (2)$$

For an initial  $x_0 \in [-1,1]$ , a numerical orbit  $x_0, x_1, \dots, x_k, \dots$  is obtained by (2), by the following coarse-grained method (3), it is transformed into a sequence  $W = s_1 s_2 \dots s_k \dots, k \in Z^+$ .

$$s_k = \begin{cases} L, & \text{if } x_k \in [-1, c); \\ C, & \text{if } x_k = c; \\ M, & \text{if } x_k \in (c, d); \\ D, & \text{if } x_k = d; \\ R, & \text{if } x_k \in (d, 1]. \end{cases} \quad (3)$$

Here,  $L, M$  and  $R$  are three monotonous limbs respectively,  $L$  and  $R$  are monotone increasing,  $M$  monotone decreasing, the case  $(+-)$  is considered, another case  $(-+)$  is trivial and omitted. We call  $W$  is a periodic sequence if  $W = (s_1 s_2 \dots s_p)^\infty$  and period is  $p = |W|$ . If

W pass through all two critical points C and D, while W is periodic, we note  $W = (XDYC)^\infty = XDYC$  by normalization to let C at the end, we call W is SSKS with single cycle. If  $W = ((YC)^\infty, (XD)^\infty)$ , W is SSKS with double cycles, it is normalized to  $W = (YC, XD)$  for conciseness, where X and Y are sequences composed of letters from {L, M, R}.

### 2.1.2. MSS Order and Admissibility Conditions

Let  $L \prec C \prec M \prec D \prec R$  be the MSS order [7], because M is the sole monotone decreasing limb, the operator  $\tau$  is defined as

$$\tau(X) = \begin{cases} 1, & \text{the number of } M \text{ in } X \text{ is even,} \\ -1, & \text{the number of } M \text{ in } X \text{ is odd.} \end{cases} \quad (4)$$

Where X is a sequence composed of letters from {L, M, R}.

The order rule between sequence W1 and W2 is described as: Let  $W_1 = \Delta a \dots$  and  $W_2 = \Delta b \dots$ ,  $\Delta$  is their common leading part and  $a \neq b$ , then if  $\tau(\Delta) = 1$  and  $a \succ b$  then  $W_1 > W_2$ , if  $\tau(\Delta) = 1$  and  $a \prec b$  then  $W_1 < W_2$ ; if  $\tau(\Delta) = -1$  and  $a \succ b$  then  $W_1 < W_2$ , if  $\tau(\Delta) = -1$  and  $a \prec b$  then  $W_1 > W_2$ .

We call all subsequences of s in W is denoted as  $\bar{s}(W)$ ,  $s \in \{L, C, M, D, R\}$ , according to the geometric meaning of  $\bar{s}(W)$ , because C is the peak and D is the vally in bimodal maps, the following admissibility conditions are easily obtained,

$$\begin{cases} \bar{L}(W) < \bar{C}(W), \bar{M}(W) < \bar{C}(W) \\ \bar{D}(W) < \bar{M}(W), \bar{D}(W) < \bar{R}(W). \end{cases} \quad (5)$$

### 2.1.3. The Production of Admissible Sets for Two Kinds of SSKS

**Table 1.** The admissible number of SSKS for period 2-15

Period n	$XDYC$	$(YC, XD)$
2	1	0
3	2	0
4	5	1
5	12	4
6	30	14
7	78	44
8	205	129
9	546	372
10	1476	1064
11	4026	3020
12	11070	8554
13	30660	24256
14	85410	68862
15	239144	195868

n-period SSKSs with single cycle is easily define as  $n = |XDYC|$ , for n-period SSKSs with double cycle  $W = (YC, XD)$  is define as  $n = |YC| + |XD|$ . By  $3^{n-2} \cdot (n-1)$  permutation and (5), all admissible SSKSs with single cycle and double cycles are produced by computer program respectively, the number of admissible words

presented in Tab. 1.

All the admissible sets provide the important SSKSs for parameters calculating, star products, equal topological entropy classification and so on.

## 2.2. Word Lifting Technique and Iterative Map for SSKS

The essence of symbolic dynamics is coarse-grained, in spite of the loss of the accuracy, however, by use of the ‘‘joints’’ property and the word lifting technique, all parameters decided by the admissible SSKS can be obtained by newton-iterative method. Here, we demonstrate the process of parameters calculating for SSKS ( $RC, LMD$ ) with double cycles.

$$\begin{cases} f(c, c, d) = f_R^{-1}(c, c, d) \\ f(d, c, d) = f_L^{-1} \circ f_M^{-1}(d, c, d) \end{cases} \quad (6)$$

Here,  $f_L^{-1}$ ,  $f_M^{-1}$  and  $f_R^{-1}$  are inverse limbs function in (1) respectively,  $\circ$  is the composite function operator, from (6), a system of nonlinear equations is written as

$$\begin{cases} F_1(c, d) = f(c, c, d) - f_R^{-1}(c, c, d) = 0 \\ F_2(c, d) = f(d, c, d) - f_L^{-1} \circ f_M^{-1}(d, c, d) = 0 \end{cases} \quad (7)$$

For an initial value  $c_0 = -0.5, d_0 = 0.5$ ,

$$J_n = \frac{\partial(F_1, F_2)}{\partial(c, d)} \Big|_{p_n} = \begin{vmatrix} \frac{\partial F_1}{\partial c} & \frac{\partial F_1}{\partial d} \\ \frac{\partial F_2}{\partial c} & \frac{\partial F_2}{\partial d} \end{vmatrix} \Big|_{p_n=(c_n, d_n)}, \quad J_n \text{ is the}$$

Jacobian determinant, (7) can be solved by the iterations (8):

$$c_{n+1} = c_n - \frac{J_n^i}{J_n}, d_{n+1} = d_n - \frac{J_n^j}{J_n}. \quad (8)$$

Where,  $J_n^i = J_n \Big|_{ith \text{ column} \rightarrow (F_1, F_2)^T}$  is the determinant

obtained from  $J_n$  by replacing its  $i$ th column by  $(F_1, F_2)^T$ .

The newton iterative makes (8) converge to a fixed point quickly. Four partial derivatives may be computed by finite difference,

$$\begin{aligned} & \text{By the taylor expansion, } F_1(c+h, d) = F_1(c, d) + \frac{\partial F_1(c, d)}{\partial c} h + \frac{\partial^2 F_1(c, d)}{\partial c^2} h^2 + o(h^2) \quad \text{and} \quad F_1(c-h, d) = \\ & F_1(c, d) + \frac{\partial F_1(c, d)}{\partial c} (-h) + \frac{\partial^2 F_1(c, d)}{\partial c^2} h^2 + o(h^2), \quad \text{two} \\ & \text{equations subtract, we get } \frac{\partial F_1}{\partial c} \approx \frac{F_1(c+h, d) - F_1(c-h, d)}{2 * h}, \end{aligned}$$

the rest partial derivatives can be obtained in the same way.

Because the IEEE double precision float has only 16 significant digits, generally,  $h = 1e-14$ .

If the accumulation error becomes larger and obtain a false solution or failure to converge into a fixed point during the Newton-iterative process, we may use high-precision library such as NTL in C++, VPA in matlab and pypmpf in Python, h takes 1e-30 for more precision of four partial derivatives. Two iterative graphs for ( $RC, LMD$ ) and  $RLDLMC$  are presented in Fig.1.

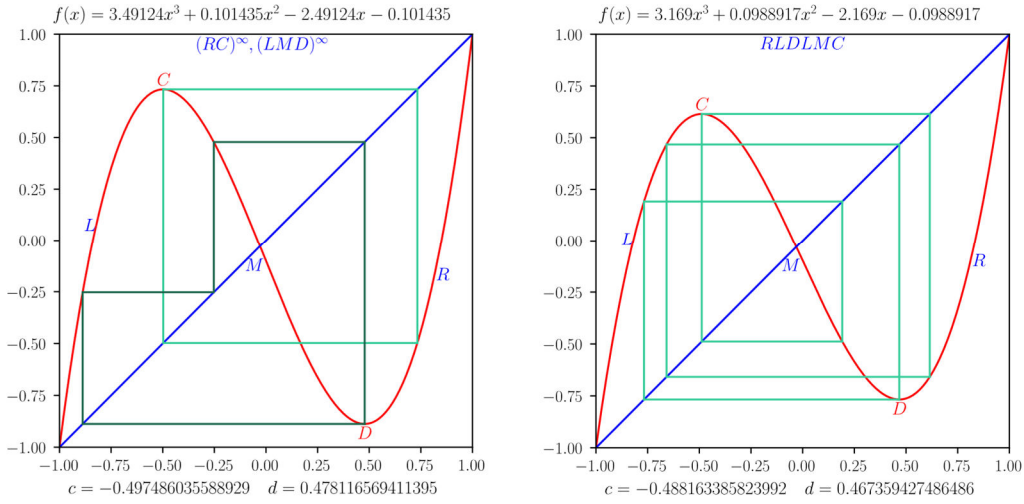


Figure 1. The iterative graphs of SSKS ( $RC, LMD$ ) and  $RDLMLC$

### 3. Parameters of Admissible SSKS with Single Cycle and Double Cycles

#### 3.1. Calculating Parameters for Every SSKS in Admissible Sets

From the admissible SSKS sets and (6)-(8) determined by SSKS, a series of python programs are coded to accomplish the task. Here, the codes of batch calculating program is omitted, a few parameters of SSKS with single cycle  $XDYC$  are calculated and presented in Tab. 2, these data are selected as example to show the algorithm works well.

Table 2. Parameters of SSKS with single cycle

$XDYC$	$c$	$d$
DC	-0.447213595	0.447213595
RDC	-0.444817339	0.509943818
RMDC	-0.4461781471	0.487777394
DLLMC	-0.514964385	0.444438629
RRDLLC	-0.499618900	0.499618900
RLLDLC	-0.513549329	0.456351149
DLLMLMLC	-0.514512706	0.444473726

While that of SSKS with double cycles ( $YC, XD$ ) presented in Tab. 3.

Table 3. Parameters of SSKS with double cycles

$(YC, XD)$	$c$	$d$
(RC, LD)	-0.475963149	0.475963149
(RRC,LD)	-0.478043114	0.504120442
(RRC, LRD)	-0.453354792	0.509840573
(RMC, LMLD)	-0.479973058	0.496948024
(RLMLC, LLD)	-0.506129847	0.470285197
(RC, LLMMRLD)	-0.505410068	0.477999656
(RMMC, LRLRMD)	-0.470854670	0.491495738
(RMLRLMC, LMD)	-0.492859555	0.494445613

All the result will provide important data for further research and applications on symbolic dynamics in bimodal maps.

#### 3.2. Scatter Plots on the Parameter Plane

There are 17446 blue points which stand for the admissible

SSKS with single cycle and 13198 red points for the admissible SSKS with double cycles ( $YC, XD$ ), see Fig2.

In fact, we have calculated the parameters of admissible SSKSs from period 2-12, it consumed cpu time about twenty hours.

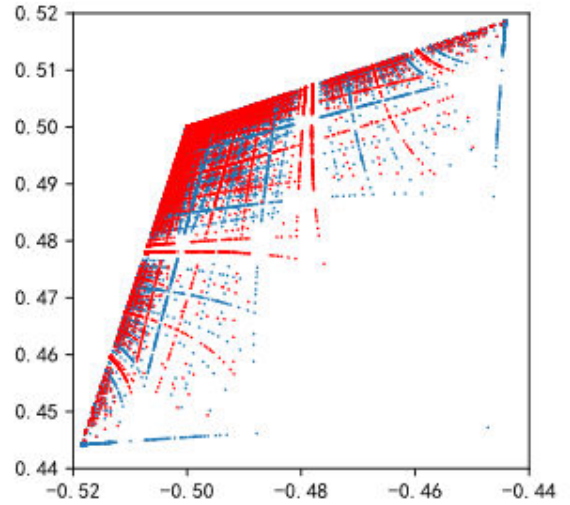


Figure 2. Scatter plots for all joints on the parameters plane

It is easy to realize there are clear boundaries on the parameter plane, outside of the boundaries may stand for the non-endomorphism map on interval  $[-1,1]$  and non-admissible sequences. How to get the boundaries is worth of thinking. On the other hand, the left-top corner gathers many points and zooming locally, the scatter plots have fractal structures apparently, the right corner stands for the DC SSKS with single cycle.

### 4. Conclusion

For bimodal maps, the admissible set of SSKS can be calculated and obtain corresponding parameter set. Admissible SSKSs with double cycles ( $YC, XD$ ) have same importance as that of single cycle. There are arising many interesting questions from the Fig.2, What is the boundary? If it is a square area? Points on the same lines

corresponds to admissible SSKSs, what properties do they have? The answer will improve the understanding of symbolic dynamics in bimodal maps and worthy of researching further.

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