Journal of Computing and Electronic Information Management
ISSN: 2413-1660 | Vol. 15, No. 1, 2024

Triply Bifurcations and Metric Universalities in 1D Trimodal Maps

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Abstract: The famous Feigenbaum's constants such as scaling factor and convergence rate can be measured and recomputed in unimodal maps, one of the methods is by the n-tupling bifurcations to chaos and parameter calculation algorithm which are called star transformation and word-lifting technique respectively. In the paper, we presented a kind of simple triply star transformation rule and the corresponding proof of admissible three super-stable kneading sequences (TSSKS), a few of TSSKS to chaos by triply bifurcations and the metric universalities are investigated.

Keywords: Symbolic dynamics; Triply bifurcation; Metric Universalities.

1. Introduction

Symbolic dynamics played an important role in the research of chaos behavior for iterative system [1]. The most key terminology is super-stable kneading sequence (SSKS), first, parameters of a SSKS is computable based on the admissibility, second, "joints" , "flesh" and "skeleton" are proposed by MacKay and Tresser [2-3], the former is just SSKS, they construct the corresponding symbolic space together, third, Feigenbaum found the metric universal constants during the doubling bifurcation for unimodal iterative system, it was well known the milestone of symbolic dynamics, however, a series of star product are presented by Hao B.L. and Peng S.L. for unimodal and bimodal maps [4-6], another way to reappear the Feigenbaum's constants, SSKS is one of the kernel.

SSKS was defined in the Hao B.L. Book [6] as: A SSKS is a kind of periodic sequence which pass through all critical points in the iterated map, so a TSSKS in trimodal maps is $(ZEXDYC)^\infty$ or $(XDZEYC)^\infty$, this kind of single-cycle SSKS is naturally researched and played a crucial role in fact. With the deep understanding of SSKS, we give an empirical and equivalent definition: Periodic sequences who can be calculated the parameters of the iterated map by the word lifting technique are called SSKS.

For 1D trimodal maps, SSKS includes TSSKS such as $(ZEXDYC)^\infty$ and $(XDZEYC)^\infty$, but also $((ZE)^\infty,(XDYC)^\infty)$, $((XD)^\infty,(ZEYC)^\infty)$ and $((YC)^\infty,(XDZE)^\infty)$, for the sake of periodicity, $(XDYC)^\infty$ is equivalent to $(YCXD)^\infty$, $(XDZE)^\infty$ to $(ZEXD)^\infty$ and so on. All SSKSs are a point or a vector of n-dimensional parameter space, the so-called "joints" in the symbolic space. The Newton iterative method is used to calculate the parameters of SSKSs.

In this paper, For an arbitrary SSKS W, we presented a triply star transformation operator (TSTO) φ, such that $\varphi(W)$ is a SSKS with same type, $\left|\varphi^k(W)\right| = 3^k\left|W\right|$, $k \in N$. $\left|W\right|$ is the period of W, φ is called triply tupling operator, its admissibility holding is proved, the generalized Feigenbaum's constants is calculated during the triply tupling bifurcation. For the need of high precision by word-lifting technique, the python library mpmath is used.

2. Symbolic Dynamics in Trimodal Maps

2.1. Iterative Maps and Symbolic Dynamics

The 1D trimodal maps are quartic maps with parameters as follows,

$$f(x,\mu) = \sum_{n=0}^{4} a_n x^n \tag{1}$$

Where, $\mu = (a_0, a_1, a_2, a_3, a_4)$, f has four explicit inverse limbs functions can be used in the word-lifting algorithm. For general algebraic maps of orders higher than or equal to 5, the word-lifting technique for calculating parameters of symbolic sequences meets with a difficulty of lacking explicit expressions of the inverse functions of the maps. An effective numerical method of the word-lifting technique has solved this problem [7]. In order to meet the requirement of word-lifting algorithm, the map f may be defined as:

$$f(x,\mu) = k\int (x-c)(x-d)(x-e)dx + i \tag{2}$$

C, *D* and *E* stands for three critical points, *c*, *d* and *e* for the corresponding horizontal coordinates respectively, also called three stationary points of f.

Consider endomorphism interval [-1,1] and two boundary conditions $f(-1) = -1$ and $f(1) = -1$, k and b are eliminated by $k = -4(1+i)/(1+2(cd+ce+be))$ and $b = -(c+e)/(1+3ce)$, the iterative equation is rewritten as (3), $\mu = (c,e,i)$.

$$x_{n+1} = f(x_n, \mu) \tag{3}$$

For an initial $x_0 \in [-1,1]$, a numerical orbit $x_0, x_1, \cdots, x_k, \cdots$ is obtained by (3), by the following so-called coarse-grained method (4), it is transformed into a sequence $W = s_0 s_1 \cdots s_k \cdots,\ \ k \in Z^+$.

$$s_k = \begin{cases} L, & if\ x_k \in [-1, c); \\ C, & if\ x_k = c; \\ M, & if\ x_k \in (c, d); \\ D, & if\ x_k = d; \\ N, & if\ x_k \in (d, e); \\ E, & if\ x_k = e; \\ R, & if\ x_k \in (e, 1]. \end{cases} \tag{4}$$

Here, L, M, N and R are four monotonous limbs respectively, L and N are monotone increasing, M and R monotone decreasing, the case (+-+-) is considered in the paper, another case (-+-+) is trivial and omitted. We call W is a periodic sequence if $W = (s_1 s_2 \cdots s_p)^\infty$ and period is $p = |W|$. If W pass through all three critical points C, D and E, while W is periodic, we note $W = (ZEXDYC)^\infty = ZEXDYC$ by normalization to let C at the end.

2.2. Admissible Conditions

Let $L \prec C \prec M \prec D \prec N \prec E \prec R$ be the MSS order, because M and R are monotone decreasing limb, the operator τ is defined as

$$\tau(X) = \begin{cases} 1, & \text{the number of } M \text{ and } R \text{ in } X \text{ is even,} \\ -1, & \text{the number of } M \text{ and } R \text{ in } X \text{ is odd.} \end{cases} \tag{5}$$

Where X is a sequence composed of letters from {L, M, N,R}.

The order rule between sequence W_1 and W_2 is described as: Let $W_1 = \Delta a \cdots$ and $W_2 = \Delta b \cdots$, Δ is their common leading part and $a \neq b$, then if $\tau(\Delta) = 1$ and $a \succ b$ then $W_1 \succ W_2$, if $\tau(\Delta) = 1$ and $a \prec b$ then $W_1 \prec W_2$; if $\tau(\Delta) = -1$ and $a \succ b$ then $W_1 \prec W_2$, if $\tau(\Delta) = -1$ and $a \prec b$ then $W_1 \succ W_2$.

We call all subsequences of s in W is denoted as $\bar{s}(W)$, $s \in \{L, C, M, D, N, E, R\}$, according to the geometric meaning of $\bar{s}(W)$, which stands for the height of $\bar{s}(W)$ in the symbolic space. Because C and E are peaks and D is a valley in trimodal maps for the case of (+-+-), the following admissibility conditions are easily obtained by Fig.1.

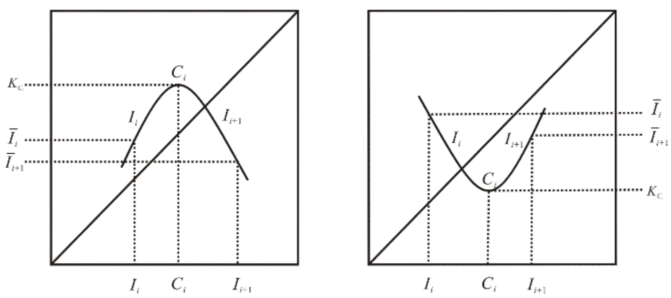

Fig.1 The schematic diagram of subsequences of peaks and valley

$$\begin{cases} \bar{C}(W) \succ \bar{L}(W), & \bar{C}(W) \succ \bar{M}(W), & C \text{ is a peak;} \\ \bar{D}(W) \prec \bar{M}(W), & \bar{D}(W) \prec \bar{N}(W), & D \text{ is a valley;} \\ \bar{E}(W) \succ N(W), & \bar{E}(W) \succ \bar{R}(W), & E \text{ is a peak.} \end{cases} \tag{6}$$

2.3. Star Products and Star Transformation

All admissible TSSKS with form $ZEXDYC$ constitute a set K. The disturbance of three critical points C, D and E are defined as follows

$$\begin{cases} C^0 = C, & C^{-1} = L, & C^{+1} = M; \\ D^0 = D, & D^{-1} = M, & D^{+1} = N; \\ E^0 = E, & E^{-1} = N, & E^{+1} = R. \end{cases} \tag{7}$$

Start product * is defined as :

$$\begin{aligned} &(ZEXDYC) * (EDC) = ZEXD^{\tau(X)} YC^{-\tau(Y)} \\ &ZE^{-\tau(Z)} XDYC^{-\tau(Y)} ZE^{-\tau(Z)} XD^{-\tau(X)} YC \end{aligned} \tag{8}$$

Here, Z, X and Y are sequences with finite length which are consisted of {L,M,N,R}.

Lemma 1.

$$\begin{cases} YC^{-\tau(Y)} \prec YC \prec YC^{\tau(Y)} \\ XD^{-\tau(X)} \prec XD \prec XD^{\tau(X)} \\ ZE^{-\tau(Z)} \prec ZE \prec ZE^{\tau(Z)} \end{cases} \text{ holds.} \tag{9}$$

Proof. C is a peak, if $\tau(Y) = 1$, $YC^{-\tau(Y)} = YL$, $YC^{\tau(Y)} = YM$, for $L < C < M$, the first inequation holds, if $\tau(Y) = -1$, $YC^{-\tau(Y)} = YM$, $YC^{\tau(Y)} = YL$, the first inequation holds too.

D is a valley, if $\tau(X) = 1$, $XD^{-\tau(Y)} = XM$, $XD^{\tau(X)} = XN$, for $M < D < N$, the second inequation holds, if $\tau(X) = -1$, $XD^{-\tau(X)} = YN$, $XD^{\tau(X)} = XM$, the second inequation holds too. The third inequation can be proofed as the same way.

Lemma 2. $\tau(ZE^{\tau(Z)} XD^{\tau(X)} YC^{\tau(Y)}) = 1$ holds.

Proof. The eight cases for $\tau(Z) = \pm 1$, $\tau(X) = \pm 1$ and $\tau(Y) = \pm 1$ are discussed respectively. Here, the first case is $\tau(Z) = \tau(X) = \tau(Y) = 1$, then $ZE^{\tau(Z)} XD^{\tau(X)} YC^{\tau(Y)} = ZRXNYM$, so $\tau(ZRXNYM) = \tau(R) \cdot \tau(N) \cdot \tau(M) = 1$ holds. Other cases may be considered the same way, omitted here for conciseness.

Lemma 3. If $\begin{cases} W * E = ZEXD^{\tau(X)} YC^{-\tau(Y)} \\ W * D = ZE^{-\tau(Z)} XDYC^{-\tau(Y)} \\ W * C = ZE^{-\tau(Z)} XD^{-\tau(X)} YC \end{cases}$ holds,

then $W * E \succ W * D \succ W * C$, $W = ZEXDYC$.

Proof. By Lemma 1, $\because ZE \succ ZE^{-\tau(Z)}$, $W * E \succ W * D$ holds apparently. Because $W * D$ and $W * C$ have the common leading string $\Delta = ZE^{-\tau(Z)} X$, if $\tau(Z) = 1$, then $ZE^{-\tau(Z)} = ZN$, so $\tau(ZE^{-\tau(Z)}) = 1$; if $\tau(Z) = -1$, then $ZE^{-\tau(Z)} = ZR$, so $\tau(ZE^{-\tau(Z)}) = \tau(Z)\cdot\tau(R) = 1$ holds too forever. If $\tau(X) = -1$, then $\tau(\Delta) = \tau(ZE^{-\tau(Z)})\cdot\tau(X) = -1$, for $D^{-\tau(X)} = D^{+1} = N$, in addition $D^{-\tau(X)} \succ D$, by the sort rule of symbolic, $W*D \succ W*C$ holds. On the other hand, If $\tau(X) = 1$, then $\tau(\Delta) = \tau(ZE^{-\tau(Z)})\cdot\tau(X) = 1$, for $D^{-\tau(X)} = D^{-1} = M$, in addition $D^{-\tau(X)} \prec D$, by the sort rule of symbolic, $W*D \succ W*C$ holds too. So, Lemma 3 is proved correct.

Theorem 1. If $W = ZEXDYC \in \mathrm{K}$, a triply star transformation operator (TSTO) φ is introduced as

$$\varphi(W) = W*(EDC) = ZEXD^{\tau(X)}YC^{-\tau(Y)}ZE^{-\tau(Z)} XDYC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YC. \quad (10)$$

A Triply bifurcations routes to chaos is obtained by $W,\ \varphi(W),\ \varphi^2(W),\cdots\varphi^k(W),\cdots$, $\varphi^0(W) = W$, $\varphi^1(W) = \varphi(W)$, $\varphi^{k+1}(W) = \varphi(\varphi^k(W)) \in \mathrm{K}$, $\left|\varphi^{k+1}(W)\right| = 3^{k+1}|W|$, $k \in N.$

Proof. From formula (10), $|\varphi(W)| = 3|W|$ obviously holds, so $\left|\varphi^{k+1}(W)\right| = 3^{k+1}|W|$ holds for $k \in N$. In fact, the key proof of theorem 1 is $\varphi^{k+1}(W) \in \mathrm{K}$, just prove $\varphi(W) \in \mathrm{K}$. Let $\varphi(W) = Z_1EX_1DY_1C$, here Z_1, X_1, Y_1, Z, X, Y is sequences composed of {L,M,N,R} respectively. In order to prove the admissibility of $\varphi(W)$, three subsequences of critical points *C*, *D* and *E* are written as

$$\begin{cases} \bar{C}(\varphi(W)) = ZE \\ \bar{D}(\varphi(W)) = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD^{-\tau(X)}YC \\ \bar{E}(\varphi(W)) = XD^{\tau(X)}YC^{-\tau(Y)}ZE^{-\tau(Z)}XD \end{cases} \quad (11)$$

It is worthy of noting that the end of subsequence of each critical point is one of other critical points, no need to write the remnant part, the real reason is that three critical points are final symbol in the process of comparison of two sequences. For example, $\bar{C}(\varphi(W)) = ZE$ is more inherent than $\bar{C}(\varphi(W)) = ZEXD^{\tau(X)}YC^{-\tau(Y)}ZE^{-\tau(Z)}\cdots$. In eq. (10), first, $\overline{D^{\tau(X)}}(\varphi(W)) = YC^{-\tau(Y)}ZE^{-\tau(Z)}XD$, if $\tau(X) = 1$, $\overline{D^{\tau(X)}}(\varphi(W)) = \bar{N}(\varphi(W))$, because $W = ZEXDYC \in \mathrm{K}$, from (6), $YC \prec XD$ holds, by Lemma 1, $YC^{-\tau(Y)} \prec YC \prec XD \prec XD^{\tau(X)}$, $\therefore \bar{D}(\varphi(W)) \prec \bar{N}(\varphi(W)) \prec \bar{E}(\varphi(W))$ is true, if $\tau(X) = -1$, $\overline{D^{\tau(X)}}(\varphi(W)) = \bar{M}(\varphi(W))$, because $W = ZEXDYC \in \mathrm{K}$, from (6), $YC \prec ZE$ holds, by Lemma 1, $YC^{-\tau(Y)} \prec YC \prec ZE \prec ZE^{\tau(Z)}$, $\therefore \bar{D}(\varphi(W)) \prec \bar{M}(\varphi(W)) \prec \bar{C}(\varphi(W))$ is true.

Second, $\overline{D^{-\tau(X)}}(\varphi(W)) = YC$, if $\tau(X) = -1$, $\overline{D^{-\tau(X)}}(\varphi(W)) = \bar{N}(\varphi(W))$, because $W = ZEXDYC \in \mathrm{K}$, from (6), $YC \prec XD$ holds, by Lemma 1, $YC^{-\tau(Y)} \prec YC \prec XD \prec XD^{\tau(X)}$, $\therefore \bar{D}(\varphi(W)) \prec \bar{N}(\varphi(W)) \prec \bar{E}(\varphi(W))$ is true, if $\tau(X) = 1$, $\overline{D^{-\tau(X)}}(\varphi(W)) = \bar{M}(\varphi(W))$, because $W = ZEXDYC \in \mathrm{K}$, from (6), $YC \prec ZE$ holds, by Lemma 1, $YC^{-\tau(Y)} \prec YC \prec ZE \prec ZE^{\tau(Z)}$, $\therefore \bar{D}(\varphi(W)) \prec \bar{M}(\varphi(W)) \prec \bar{C}(\varphi(W))$ is true.

In eq. (10), the perturbations of critical point E have two forms $\overline{E^{-\tau(Z)}}(\varphi(W)) = XD$ and $\overline{E^{-\tau(Z)}}(\varphi(W)) = XD^{-\tau(X)}YC$. Only the former is proved, the latter can be proved same way, here is omitted.

If $\tau(Z) = -1$, $\overline{E^{-\tau(Z)}}(\varphi(W)) = \bar{R}(\varphi(W))$, by Lemma 1, $XD \prec XD^{\tau(X)}$, $\therefore \bar{R}(\varphi(W)) \prec \bar{E}(\varphi(W))$ holds, if $\tau(Z) = 1$, $\overline{E^{-\tau(Z)}}(\varphi(W)) = \bar{N}(\varphi(W))$, because $W = ZEXDYC \in \mathrm{K}$, from (6), $YC \prec ZE$ holds, by lemma 1, $YC^{-\tau(Y)} \prec YC \prec ZE \prec ZE^{\tau(Z)}$, $\therefore \bar{D}(\varphi(W)) \prec \bar{N}(\varphi(W)) \prec \bar{E}(\varphi(W))$ holds.

Third, in eq. (10), the perturbations of critical point C have two forms $\overline{C^{-\tau(Y)}}(\varphi(W)) = ZE^{-\tau(Z)}XD$ and $\overline{C^{-\tau(Y)}}(\varphi(W)) = ZE^{-\tau(Z)}XD^{-\tau(X)}YC$. Only the former is proved, the latter can be proved same way, here is omitted.

If $\tau(Z) = -1$, $\overline{C^{-\tau(Y)}}(\varphi(W)) = \bar{M}(\varphi(W))$, by admi admissible conditions of *W* in (6), $ZE \prec XD$ holds, by lemma 1, $ZE^{-\tau(Z)} \prec ZE \prec XD \prec XD^{\tau(Z)}$ is true, so $\bar{D}(\varphi(W)) \prec \bar{M}(\varphi(W)) \prec \bar{C}(\varphi(W))$ holds; If $\tau(Z) = 1$, $\overline{C^{-\tau(Y)}}(\varphi(W)) = \bar{R}(\varphi(W))$, because $W = ZEXDYC \in \mathrm{K}$, from (6), $YC \prec ZE$ holds, by Lemma 1, $YC^{-\tau(Y)} \prec YC \prec ZE \prec ZE^{\tau(Z)}$, $\therefore \bar{D}(\varphi(W)) \prec \bar{N}(\varphi(W))$ holds, by lemma 1, $ZE^{-\tau(Z)} \prec ZE$, $\therefore \therefore \bar{D}(\varphi(W)) \prec \bar{N}(\varphi(W)) \prec \bar{E}(\varphi(W))$ holds at the same time. So, subsequences of disturbance of the eight critical points in (10) satisfy the admissible conditions (6).

Final, the original letters from {L,M,N,R} in *W* are still preserved in $\varphi(W)$, because $W = ZEXDYC \in \mathrm{K}$, from (6), for example, $\bar{L}(W) \prec \bar{C}(W)$, by lemma 3, $\varphi(\bar{L}(W)) \prec \varphi(\bar{C}(W))$, $\therefore \bar{L}(\varphi(W)) \prec \bar{C}(\varphi(W))$, similarly, $\bar{D}(\varphi(W)) \prec \bar{M}(\varphi(W)) \prec \bar{C}(\varphi(W))$, $\bar{D}(\varphi(W)) \prec \bar{N}(\varphi(W)) \prec \bar{E}(\varphi(W))$ and

$\bar{E}(\varphi(W)) \prec \bar{R}(\varphi(W))$ hold together, $\therefore \varphi(W)$ satisfies the admissible conditions (6), $\varphi(W) \in \mathrm{K}$, so $\varphi^{k+1}(W) \in \mathrm{K}$, for $k \in N$. The proof is over.

Corollary 1. If $W = ZEXDYC \in \mathrm{K}$, then the admissibility decides the TSSKS W corresponds to a real iterative orbit, the parameters of W are computable.

Remak: The result shows a triply tupling bifurcation may be metrically calculated.

2.4. Calculation of parameters for TSSKS

The iterative maps should use (2) and (3). The essence of symbolic dynamics is coarse-grained, in spite of the loss of the accuracy, however, by use of the "joints" property and the word lifting technique, all parameters decided by the admissible SSKS can be obtained by newton-iterative method. Here, we demonstrate the process of parameters calculating for a TSSKS like *ZEXDYC*, for instance, *ERNDC* corresponds a system of nonlinear equations

$$\begin{cases} F_1(c,d,e) = f(c,c,d,e) - e = 0 \\ F_2(c,d,e) = f(e,c,d,e) - f_R^{-1} \circ f_N^{-1}(d,c,d,e) = 0 \\ F_3(c,d,e) = f(d,c,d,e) - c = 0 \end{cases} \quad (12)$$

Here, f_R^{-1} and f_N^{-1} are inverse limbs function in (2) respectively, $\circ$ is the composite function operator.

For an initial value $c_0 = -0.61, d_0 = 0.02, e_0 = 0.7$,

$$J_n = \left.\frac{\partial(F_1,F_2,F_3)}{\partial(c,d,e)}\right|_{P_n} = \left.\begin{vmatrix} \frac{\partial F_1}{\partial c} & \frac{\partial F_1}{\partial d} & \frac{\partial F_1}{\partial e} \\ \frac{\partial F_2}{\partial c} & \frac{\partial F_2}{\partial d} & \frac{\partial F_3}{\partial e} \\ \frac{\partial F_3}{\partial c} & \frac{\partial F_3}{\partial d} & \frac{\partial F_3}{\partial e} \end{vmatrix}\right|_{P_n=(c_n,d_n,e_n)},$$

J_n is the Jacobian determinant, (12) can be solved by the iterations.

$$c_{n+1} = c_n - \frac{J_n^i}{J_n}, d_{n+1} = d_n - \frac{J_n^i}{J_n}, e_{n+1} = e_n - \frac{J_n^i}{J_n}. \quad (13)$$

Where, $J_n^i = J_n|_{ith\ column \to (F_1,F_2,F_3)^T}$ is the determinant obtained from J_n by replacing its *ith* column by $(F_1,F_2,F_3)^T$. The newton iterative makes (13) converge to a fixed point quickly. Nine partial derivatives may be computed by finite difference method.

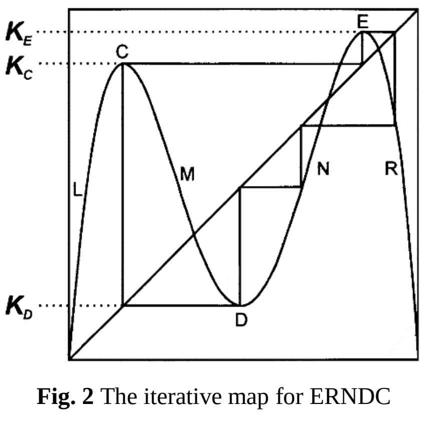

Fig. 2 The iterative map for ERNDC

Fig.2 shows that the word-lifting technique is useful and the metric calculation in the triply tupling bifurcation process may be explored furtherly.

3. Metric Universalities in the Triply Tupling Bifurcation

If $W \in \mathrm{K}$, the TSTO φ will determine a route to chaos by triply tupling bifurcation, W, $\varphi(W)$, $\varphi^2(W)$, $\cdots$, $\varphi^k(W), \cdots, k \in N$.

A series of parameters $\mu_k = (c_k, d_k, e_k)$ of W can be calculated by the word-lifting technique. One of the generalized Feigenbaum's constants [8], the famous convergent rates are defined as

$$\delta_{\mu,k} = \frac{\mu_{k-1} - \mu_{k-2}}{\mu_k - \mu_{k-1}} \quad (13)$$

In Tab.1, we take $W = EDC$, three convergence rates are given by $\mu_k = (c_k, d_k, e_k)$ and eq. (13).

Table 1. Three convergence rates from $\varphi^k(EDC)$

k	$\delta_{c,k}$	$\delta_{d,k}$	$\delta_{e,k}$
4	12.073	11.141	4.485
5	13.032	12.798	11.367
6	13.341	13.278	12.909
7	13.427	13.410	13.310
8	13.454	13.449	13.421

In Tab.2, we takes $W = ERNDC$, three convergence rates are given by $\mu_k = (c_k, d_k, e_k)$ and eq. (13).

Table 2. Three convergence rates from $\varphi^k(ERNDC)$

k	$\delta_{c,k}$	$\delta_{d,k}$	$\delta_{e,k}$
4	13.886	14.070	15.146
5	13.629	13.681	13.982
6	13.508	13.522	13.605
7	13.479	13.483	13.506
8	13.439	13.470	13.421

In Tab.3, we takes $W = EMDC$, three convergence rates are given by $\mu_k = (c_k, d_k, e_k)$ and eq. (13).

Table 3. Three convergence rates from $\varphi^k(EMDC)$

k	$\delta_{c,k}$	$\delta_{d,k}$	$\delta_{e,k}$
4	12.460	12.438	11.608
5	13.245	13.233	13.102
6	13.369	13.365	13.342
7	13.441	13.441	13.435
8	13.458	13.457	13.456

From the result presented in Tab. 1-3, it implies that the convergence rates are universal constants and only related with TSTO φ which is explained in eq. (8) and defined in theorem 1. In fact, convergence rate is one of metric universalities, others such as scale factor we do not considered in the paper. It will provide important data for further research and applications on symbolic dynamics in trimodal maps.

4. Conclusion

In the paper, for symbolic dynamics of 1D trimodal maps, a kind of TSTO φ is introduced to accomplish the triply tupling bifurcation process for a TSSKS *W*. A theorem is proposed and the admissibility of $\varphi^k(W)$ for $k \in N$ is proved. By the word-lifting technique, the parameters of an admissible word *W* in K are computable, the sole case which do not attain the fixed point during the newton iteration algorithm may be aroused by an inappropriate initial value or lower calculation precision, trying other initial values and using the famous python mpmath library are good ideas. In the Sec. 3, for W takes *EDC*, *ERNDC* and *EMDC*, three convergence rates by $\varphi^k(W)$ are calculated and presented in tab. 1-3 respectively. The result will improve the understanding of symbolic dynamics in trimodal maps and worthy of researching further.

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