

Applications of Equivalence Relations in Mathematics

Xiaolei Jiang

School of Control and Computer Engineering, North China Electric Power University, Baoding 071003, China

Abstract: To help teaching of equivalence relations in discrete mathematics course, typical applications of equivalence relations in mathematics are explored fully and clarified elaborately. A variety of applications are grouped into three types, including viewing apparently different objects as the same one, grouping different objects into one class, and defining new mathematical objects, and each type is exemplified by specific instances. The emphasis is placed on intuition, roadmap and rigor, rather than technical details.

Keywords: Equivalence Relations; Discrete Mathematics; Partitions; Equivalence Classes.

1. Introduction

Equivalence relations are reflexive, symmetric, transitive relations, which are important mathematical concepts not only in discrete mathematics, but also for the whole of mathematics [1]. Students often find difficulty in learning equivalence relations, with the reason lying partly in the definition of equivalence relations is somewhat abstract and lack of applications from the point of view of students [2].

As a matter of fact, equivalence relations are ubiquitous in mathematics, so there are a variety of applications. However, it is not an easy task to enumerate all the occurrences of equivalence relations in mathematics and it is inevitable to miss some important ones. It is the aim of this article to collect typical applications of equivalence relations and to divide them into several different categories, thus helping students appreciate the essence of equivalence relations.

By and large, we can formulate an intuitive notion into a rigorous mathematical concept by way of equivalence relations. According to the level of complexity, we summarize applications of equivalence relations into three categories: viewing apparently different objects as the same one, grouping different objects into one class, and defining new mathematical objects, each of which is exemplified in the following sections.

2. Viewing Apparently Different Objects as the Same One

If we do not want to distinguish the difference between two objects, then we can define appropriate equivalence relation, thus different objects can be equivalent, i.e., viewing them as the same one in the sense of equivalence.

2.1. Vectors in Geometry

In geometry, we often view two vectors with the same direction and the same length as one vector, irrespective of their starting points. An example is shown in Figure 1, where the lengths of AB and CD are equal and AB is parallel to CD , so \overrightarrow{AB} and \overrightarrow{CD} can be viewed as one vector.

In terms of equivalence relation, we can formulate the above as follows.

Example 1 Two vectors \overrightarrow{AB} and \overrightarrow{CD} are equivalent if $x_A - x_B = x_C - x_D$, $y_A - y_B = y_C - y_D$, where we use x and y to denote the coordinates of a point, that is to say, (x_A, y_A) is the coordinates of point A , and so on. Note that

\overrightarrow{AB} and \overrightarrow{CD} are different entities and when we say they are the same one what we mean is that they are equivalent, i.e., \overrightarrow{AB} is related to \overrightarrow{CD} by this equivalence relation.

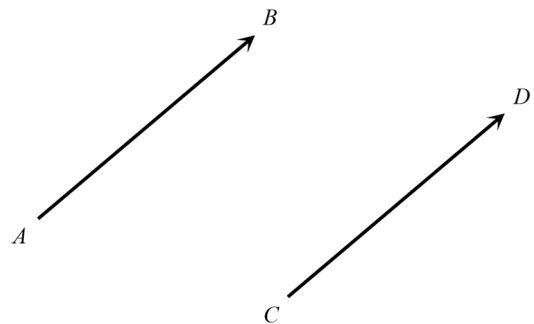


Figure 1. Two equivalent vectors in geometry

2.2. Forces in Mechanics

In theoretical mechanics, two forces lying on the same straight line with the same magnitude lead to the same effect on a rigid body [3]. An example is shown in Figure 2, where the lengths of AB and CD are equal and A, B, C, D are collinear, so \overrightarrow{AB} and \overrightarrow{CD} can be viewed as one force.

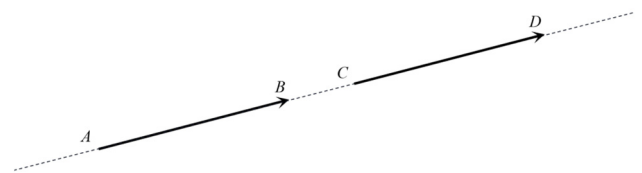


Figure 2. Two equivalent forces in mechanics

In terms of equivalence relation, we can formulate the above as follows.

Example 2 Two vectors \overrightarrow{AB} and \overrightarrow{CD} are equivalent if $x_A - x_B = x_C - x_D$, $y_A - y_B = y_C - y_D$, and

$$\begin{bmatrix} 1 & x_A & y_A \\ 1 & x_B & y_B \\ 1 & x_C & y_C \end{bmatrix} = 0. \quad (1)$$

2.3. Isomorphic Graphs

In graph theory, two apparently different graphs but sharing the common connectedness relationships between vertices are viewed as the same graph. The two graphs shown in Figure 3 can be seen as the same one, since they have the same

structure when the identities of vertices are removed, as can be shown after introducing a one-to-one correspondence between their sets of vertices: $A \leftrightarrow a, B \leftrightarrow b, C \leftrightarrow c, D \leftrightarrow d$.

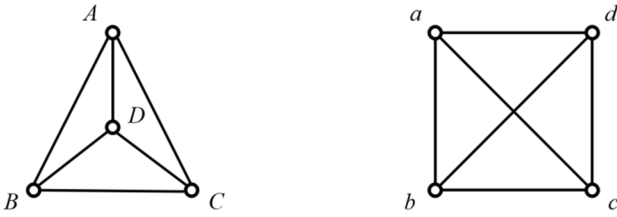


Figure 3. Two isomorphic graphs

In terms of equivalence relation, we can formulate the above as follows.

Example 3 Graph $\langle V_1, E_1 \rangle$ and graph $\langle V_2, E_2 \rangle$ are equivalent if there exists a bijection $\theta : V_1 \rightarrow V_2$ such that $(u, v) \in E_1$ if and only if $(\theta(u), \theta(v)) \in E_2$ for any $u, v \in V_1$.

2.4. Functions in Real Analysis

In real analysis, two functions differing only on a set of measure zero are often viewed as the same function. In terms of equivalence relation, we can formulate the above as follows.

Example 4 Two functions f and g are equivalent if $f - g$ is not equal to zero only on a set of measure zero. In real analysis, we say f and g are equal almost everywhere.

3. Grouping Different Objects into One Class

If the difference between two objects is so large that they cannot be viewed as the same one but is not too large to be classified into different types, then we can define appropriate equivalence relation to grouping them into the same class. This type of example for equivalence relation resembles that discussed in the last section, with the difference lying in that here we admit the difference between two objects and viewing them as different objects but classifying them into the same class, while in the last section we overlook the difference between two objects and viewing them as the same one.

In the formulation of equivalence relations, different objects can be classified into one class if and only if they are equivalent, different objects classified into one class are said to be in one equivalence class, and all equivalence classes form a partition. We often choose a representative from an equivalence class, and different objects belonging to one equivalence class can often be changed from one to another by some standard transformations. There often exists invariants which are the same for objects in one equivalence class. By an equivalence relation and the induced partition, we can better understand the structure of the set of all objects under consideration.

3.1. Matrices in Linear Algebra

Example 5 Two matrices A and B in $\mathbb{R}^{m \times n}$ are equivalent if there exists invertible matrices P and Q such that $A = PBQ$. By this equivalence relation the set of all matrices with the same shape is structured into a partition containing several equivalence classes. Elementary row and column operations can change one matrix into another matrix among one equivalence class. The rank of a matrix is an invariant for equivalent matrices. A representative of each equivalence

class is the reduced row echelon form, e.g., $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a representative of equivalence class of rank 2 among all 3 by 3 matrices.

The similarity relation and congruence relation are also equivalence relations for matrices. For similarity relation, traces and determinants are invariants by which we can easily draw a conclusion like that these two matrices are not similar because their traces are different.

3.2. Connected Vertices in a Graph

In graph theory, two vertices are in the same connected component if we can start from one vertex, go through edges and vertices, and end with another vertex. In terms of equivalence relation, we can formulate the above as follows.

Example 6 Two vertices in a graph are equivalent if there exists a path between these them. By this equivalence relation, the set of all vertices is partitioned into several equivalence classes, each of which spans a maximal connected subgraph [4].

3.3. Elements in a Group

In abstract algebra, to understand the structure of a group, the following equivalence relation is introduced.

Example 7 Let $\langle H, * \rangle$ be a subgroup of $\langle G, * \rangle$, $a, b \in G$ are equivalent if $a^{-1} * b \in H$. By this equivalence relation, a group is partitioned into equivalence classes, each of which has the same potential, thus the potential of a finite group is divisible by the potential of its subgroup.

4. Defining New Mathematical Objects

If a mathematical operation is not defined for two objects, then we can define new mathematical objects by equivalence relation to generalize the original operation. In general, the original object and the newly defined one, i.e., the equivalence class can have very different properties. It should be emphasized that strictly speaking, we view the set of all equivalent objects, i.e., equivalence class, as a new entity, and the operation is defined for the equivalence classes, rather than for the original objects.

4.1. Defining Integers from Natural Numbers

While the sum of any two natural numbers are also natural numbers, it is not the case that the set of natural numbers is closed under subtraction. To allow subtraction of any two natural numbers, that is to say, to generalize from natural numbers to integers, we can introduce new mathematical objects by the following equivalence relation.

Example 8 Let a, b, c, d be natural numbers, $\langle a, b \rangle$ and $\langle c, d \rangle$ are equivalent if $a + d = b + c$. In terms of this equivalence relation, $1 - 2$ is identified with the equivalence class of $\langle 1, 2 \rangle$.

4.2. Defining Real Numbers from Rational Numbers

The operation of limit is not closed for sequences of rational numbers. For example, the Cauchy sequence

$$1, 1.4, 1.41, 1.414, \dots$$

composed exclusively of rational numbers, but it converges to the irrational number $\sqrt{2}$. One method of defining real numbers from rational numbers is via the following

equivalence relation.

Example 9 Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be two Cauchy sequences of rational numbers, $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are equivalent if $\lim_{n \rightarrow \infty} a_n - b_n = 0$. In terms of this equivalence relation, we can say that the sequence 1, 1.4, 1.41, 1.414, ... converges to a mathematical object, which is its equivalence class.

4.3. Defining Generalized Functions from Ordinary Functions

There does not exist such an ordinary function f that the convolution of f with any ordinary function g results in g . To introduce a new mathematical object which can behave as the identity element for convolution, the following equivalence relation is needed.

Example 10 Let f_1, g_1, f_2, g_2 be integrable functions, $\langle f_1, g_1 \rangle$ and $\langle f_2, g_2 \rangle$ are equivalent if $f_1 * g_2 = g_1 * f_2$, where $*$ denotes convolution. In this way, the impulse function is the equivalence class of $\langle f, f \rangle$ [3].

5. Conclusion

Three kinds applications of equivalence relations are illustrated, with ten specific examples fully explained. Further applications of equivalence relations in mathematics are ubiquitous and cannot be fully covered in this single article. We hope that both teachers and students will benefit from our approach of exemplification in studying equivalence relations.

For teachers, illustrated by these three kinds of applications, they can organize their teaching architecture better. For students, by studying these three kinds of applications thoroughly and filling in the missing details, they can fully comprehend the idea of equivalence relations and have a clear understanding about the role played by equivalence relations in mathematics.

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