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Fourier Transforms in Digital Image Processing Courses

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Abstract: To help teaching and learning of Fourier transforms in digital image processing courses, an approach to the Fourier transforms from a standpoint of linear algebra is presented. After representing a periodic sequence by a circulant matrix, the periodic convolution can be formulated in terms of matrix multiplication, and finally, the Fourier transform is written in a form of matrix diagonalization. As a comparison, summation formulas are prevalent in traditional courses on signals and systems and matrices are scarcely found in textbooks of digital signal processing. We hope that this approach will make the Fourier transform easier to teach and learn in digital image processing courses.

Keywords: Fourier Transform; Circulant Matrix; Periodic Convolution.

1. Introduction

The Fourier transform is usually taught in a signals and systems course or a digital signal processing course [1,2,3,4], and it is also an important topic in digital image processing courses [5]. But the emphasis and perspective in these courses are different. In a signals and systems course, four kinds of Fourier representations, including continuous-time Fourier transform, continuous-time Fourier series, discrete-time Fourier transform, and discrete-time Fourier series are all discussed and studied thoroughly, while in a digital signal processing course more emphasis is placed on discrete-time Fourier transform and series. The discrete-space Fourier series, or 2D DFT is the most relevant to a digital image processing course. In addition, a digital image processing course focuses on two-dimensional transforms while signals and systems courses and digital signal processing courses only consider one-dimensional transforms. Consequently, teaching the Fourier transform in digital image processing courses is different from teaching it in signals and systems courses and digital signal processing courses.

In this article, an approach to the Fourier transform for digital image processing courses is presented. Among the many properties of Fourier transform, the most relevant to digital image processing perhaps is the Parseval's theorem and the convolution theorem, so periodic convolution must be emphasized. In signals and systems courses and digital signal processing courses, matrix algebra is seldom used, but linear algebra must have its place in a digital image processing course since the natural representation of a digital image is an array. In light of these considerations, we borrow the approach used in [6], in which the Fourier transform is discussed from the angle of circulant matrices.

The rest of this article is organized as follows. In Section 2 one-dimensional periodic sequences are considered since they are the basis for understanding periodic convolution and discrete Fourier transform. The emphasis is on the circulant matrix representation of a periodic sequence. One-dimensional periodic convolution is presented in Section 3, focusing on the fact that the convolution sum changes into matrix multiplication for the circulant matrix representation. Building upon Section 2 and Section 3, the one-dimensional Fourier transform is given in Section 4. Instead of the summation formulas in signals and systems courses and digital signal processing courses, the Fourier transform are represented from an viewpoint of matrix. Section 5 gives a similar matrix-based exposition of the two-dimensional Fourier transform through doubly block circulant matrices and the Kronecker product. We draw our conclusion in Section 6.

2. One-dimensional Periodic Sequences

A periodic sequence is a sequence $\{x_n\}_{n=-\infty}^{\infty}$ such that $x_{n+N} = x_n$, where N is a positive integer called the period. A periodic sequence can be represented by a column vector

$$[x] = [x_0 \quad \cdots \quad x_{N-1}]^{\mathrm{T}}, \tag{1}$$

or a circulant matrix

$$\mathbf{X} = \begin{bmatrix} x_0 & \cdots & x_1 \\ \vdots & \ddots & \vdots \\ x_{N-1} & \cdots & x_0 \end{bmatrix} \tag{2}$$

Note that the first column of $\mathbf{X}$ is $[x]$. A circulant matrix is a square matrix such that the second column is the circularly shifted version of the first column, the third column is the circularly shifted version of the second column, so on and so forth. From either the column vector representation $[x]$ or the circulant matrix representation $\mathbf{X}$ we can recover the original periodic sequence $\{x_n\}_{n=-\infty}^{\infty}$. The relation between a periodic sequence $\{x_n\}_{n=-\infty}^{\infty}$ and its circulant matrix representation $\mathbf{X}$ can be written in a formula:

$$\mathbf{X} = \sum_{n=0}^{N-1} x_n \mathbf{P}_N^n, \tag{3}$$

where $\mathbf{P}_N = \begin{bmatrix} 0 & 1 \\ \mathbf{I}_{N-1} & 0 \end{bmatrix}$, here $\mathbf{I}_{N-1}$ is the identity matrix of order $N-1$. In a word, there is a bijection between periodic sequences and circulant matrices, and we can switch from one to the other back and forth whenever necessary.

3. One-dimensional Periodic Convolution

For the set of periodic sequences with the same period, we can define some operations which receive one or two periodic sequences as input and return another periodic sequence as output. Given a periodic sequence x and a real number c, the scalar multiple cx of x by c is another periodic sequence obtained by multiplying each component of x by c. Given two periodic sequences x and h, their sum $x+h$ is another periodic sequence obtained by adding corresponding components of x and h.

The most important operation is the periodic convolution. Suppose x and h are periodic sequences with the same period N, then their periodic convolution, denoted as $x \circledast h$, is defined to be another periodic sequence y, where $y_n = \sum_{k=0}^{N-1} x_k h_{n-k}$. The periodic convolution satisfies the following properties. $x \circledast h = h \circledast x$, $(x \circledast h) \circledast g = x \circledast (h \circledast g)$, $x \circledast (h+g) = x \circledast h + x \circledast g$, $x \circledast (ch) = (cx) \circledast h = c(x \circledast h)$, where c is a real number.

The periodic convolution can be written in terms of matrix multiplication. Let the circulant matrix representation of x be $\mathbf{X}$, and the column vector representation of h be $[h]$, then the column vector representation of $x \circledast h$ is $\mathbf{X}[h]$. Let the circulant matrix representations of x and h be $\mathbf{X}$ and $\mathbf{H}$ respectively, then the circulant matrix representation of $x \circledast h$ is $\mathbf{XH}$. In a word, the set of periodic sequences equipped with periodic convolution is isomorphic to the set of circulant matrices equipped with matrix multiplication. As a result, the properties of matrix multiplication is shared with periodic convolution. For example, $\mathbf{XH} \neq \mathbf{HX}$ for most matrices $\mathbf{X}$ and $\mathbf{H}$, but if they are both circulant then $\mathbf{XH} = \mathbf{HX}$, since the periodic sequence corresponding to $\mathbf{X}$ commute with the periodic sequence corresponding to $\mathbf{H}$.

4. One-dimensional Fourier Transforms

It can be verified that w_N^{-k} is an eigenvalue of $\mathbf{P}_N$, with the associated eigenvector being $\left(w_N^{0k}, w_N^{1k}, \cdots, w_N^{(N-1)k}\right)$, where k is an integer, $w_N = e^{j\frac{2\pi}{N}}$, $j^2 = -1$. It should be emphasized that there are N eigenvalues of $\mathbf{P}_N$, since $w_N^{-(k+N)} = w_N^{-k}$ as a result of $w_N^N = 1$. From this fact and Equation (3) it follows that $\hat{x}_k$ is an eigenvalue of a circulant matrix $\mathbf{X}$, with the associated eigenvector being $\left(w_N^{0k}, w_N^{1k}, \cdots, w_N^{(N-1)k}\right)$, where

$$\hat{x}_k = \sum_{n=0}^{N-1} x_n w_N^{-nk}. \tag{4}$$

Now we can say that the Fourier transform of a periodic sequence x is another periodic sequence $\hat{x}$ obtained from the eigenvalue of the circulant matrix representation of x. Note that Equation (4) gives the forward transform from x to $\hat{x}$, where k ranges over all integers. Since $\hat{x}$ is also a periodic sequence, it has a column vector representation $[\hat{x}]$. Equation (4) may be rewritten in a form of matrix multiplication:

$$[\hat{x}] = \overline{\mathbf{F}_N}[x], \tag{5}$$

where $\mathbf{F}_N$ is called the Fourier matrix, whose (n,k)-entry is w_N^{nk}. Note that here n and k both start from 0. Since $\mathbf{F}_N^{\mathrm{T}} = \mathbf{F}_N$ and $\mathbf{F}_N \mathbf{F}_N^{\mathrm{H}} = N\mathbf{I}_N$,

$$[x] = \frac{1}{N}\mathbf{F}_N[\hat{x}]. \tag{6}$$

Rewrite Equation (6) in terms of components, we have the inverse transform from $\hat{x}$ to x represented in terms of components:

$$x_n = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}_k w_N^{nk}. \tag{7}$$

From Equations (5) the Parseval's theorem can be proved directly: $\sum_{k=0}^{N-1}|\hat{x}_k|^2 = [\hat{x}]^{\mathrm{H}}[\hat{x}] = [x]^{\mathrm{H}}\mathbf{F}_N^{\mathrm{T}}\overline{\mathbf{F}_N}[x] = [x]^{\mathrm{H}} N\mathbf{I}_N[x] = N[x]^{\mathrm{H}}[x] = N\sum_{n=0}^{N-1}|x_n|^2$. The Fourier transform can also be written in a form of diagonalization:

$$\mathrm{diag}([\hat{x}]) = \mathbf{F}_N^{-1}\mathbf{X}\mathbf{F}_N, \tag{8}$$

$$\mathbf{X} = \mathbf{F}_N \mathrm{diag}([\hat{x}])\mathbf{F}_N^{-1}, \tag{9}$$

where $\mathrm{diag}(\mathbf{a})$ is the square diagonal matrix with the elements of vector $\mathbf{a}$ on the main diagonal. Let x and h be two periodic sequences, $y = x \circledast h$.

then $\mathbf{Y} = \mathbf{XH} = \mathbf{F}_N \mathrm{diag}([\hat{x}])\mathbf{F}_N^{-1}\mathbf{F}_N \mathrm{diag}([\hat{h}])\mathbf{F}_N^{-1} = \mathbf{F}_N \mathrm{diag}([\hat{x}])\mathrm{diag}([\hat{h}])\mathbf{F}_N^{-1}$, so $\mathrm{diag}([\hat{y}]) = \mathrm{diag}([\hat{x}])\mathrm{diag}([\hat{h}])$. This is just the convolution theorem. In a word, the Fourier transform is the eigenvalues of a circulant matrix.

5. Two-dimensional Fourier Transforms

The two-dimensional Fourier transform follows the same line with the one-dimensional Fourier transform. Let's repeat what we have done from Section 2 to Section 4, but this time focusing on the case of two dimensions.

A two-dimensional periodic sequence is a sequence $\{x_{n_1,n_2}\}_{n_1,n_2=-\infty}^{\infty}$ such that $x_{n_1+N_1,n_2+N_2} = x_{n_1,n_2}$, where N_1 and N_2 are positive integers called the periods. The column vector form of x is $\mathrm{vec}([x])$, where $\mathrm{vec}(\mathbf{A})$ is a column vector obtained by vertically stacking the columns of the matrix $\mathbf{A}$ and

$$[x] = \begin{bmatrix} x_{0,0} & \cdots & x_{0,N_2-1} \\ \vdots & \ddots & \vdots \\ x_{N_1-1,0} & \cdots & x_{N_1-1,N_2-1} \end{bmatrix} \tag{10}$$

Note that the same notation $[x]$ stands for the sequence x in a whole period, so for a one-dimensional sequence it is a vector while for a two-dimensional sequence it is a matrix. The matrix representation $\mathbf{X}$ of x is a doubly block circulant matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_0 & \cdots & \mathbf{X}_1 \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{N_2-1} & \cdots & \mathbf{X}_0 \end{bmatrix} \tag{11}$$

where $\mathbf{X}_{n_2} = \begin{bmatrix} x_{0,n_2} & \cdots & x_{1,n_2} \\ \vdots & \ddots & \vdots \\ x_{N_1-1,n_2} & \cdots & x_{0,n_2} \end{bmatrix}$. The two-dimensional version of Equation (3) is

$$\mathbf{X} = \sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1} x_{n_1,n_2}\left(\mathbf{P}_{N_2}^{n_2} \otimes \mathbf{P}_{N_1}^{n_1}\right) \tag{12}$$

where $\otimes$ denotes the Kronecker product.

If x and h are periodic sequences sharing the same periods N_1 and N_2, then their periodic convolution, denoted as $x \circledast h$, is defined to be another periodic sequence y, where $y_{n_1,n_2} = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} x_{k_1,k_2} h_{n_1-k_1,n_2-k_2}$. The two-dimensional periodic convolution can also be represented in terms of matrix multiplication. If $y = x \circledast h$, then $\mathrm{vec}([y]) = \mathbf{X}\mathrm{vec}([h])$ and $\mathbf{Y} = \mathbf{XH}$. Analogous to the one-dimensional case, there is a bijection between the world of periodic sequences and the world of doubly block circulant matrices, and this bijection maps periodic convolution to matrix multiplication.

Similar to the one-dimensional case, $w_{N_1}^{-k_1} w_{N_2}^{-k_2}$ is an eigenvalue of $\mathbf{P}_{N_2} \otimes \mathbf{P}_{N_1}$, with the associated eigenvector being $\left(w_{N_2}^{0k_2}, w_{N_2}^{1k_2}, \cdots, w_{N_2}^{(N_2-1)k_2}\right) \otimes \left(w_{N_1}^{0k_1}, w_{N_1}^{1k_1}, \cdots, w_{N_1}^{(N_1-1)k_1}\right)$, where k_1 and k_2 are integers. From this fact and Equation (12) it follows that $\hat{x}_{k_1,k_2}$ is an eigenvalue of a circulant matrix $\mathbf{X}$, with the associated eigenvector being

$\left(w_{N_2}^{0k_2}, w_{N_2}^{1k_2}, \cdots, w_{N_2}^{(N_2-1)k_2}\right) \otimes \left(w_{N_1}^{0k_1}, w_{N_1}^{1k_1}, \cdots, w_{N_1}^{(N_1-1)k_1}\right)$

where

$$\hat{x}_{k_1,k_2} = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} w_{N_1}^{-n_1 k_1} w_{N_2}^{-n_2 k_2} \tag{13}$$

Here the eigenvalue $\hat{x}_{k_1,k_2}$ is also called the Fourier transform of x_{n_1,n_2}. Letting the k_1 and k_2 take values in the set of all integers, we can say that the Fourier transform of a periodic sequence is another periodic sequence with the same periods.

Equation (13) may be rewritten in terms of matrix multiplication as

$$[\hat{x}] = \overline{\mathbf{F}_{N_1}} [x] \overline{\mathbf{F}_{N_2}} \tag{14}$$

It follows from Equation (14) that

$$[x] = \frac{1}{N_1 N_2} \mathbf{F}_{N_1} [\hat{x}] \mathbf{F}_{N_2} \tag{15}$$

Rewrite Equation (15) in terms of components, we have the inverse transform from $\hat{x}$ to x represented in terms of components:

$$x_{n_1,n_2} = \frac{1}{N_1 N_2} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \hat{x}_{k_1,k_2} w_{N_1}^{n_1 k_1} w_{N_2}^{n_1 k_2} \tag{16}$$

The two-dimensional Fourier transform can also be written in a form of matrix multiplied by vector:

$$\mathrm{vec}([\hat{x}]) = (\overline{\mathbf{F}_{N_2}} \otimes \overline{\mathbf{F}_{N_1}}) \mathrm{vec}([x]) \tag{17}$$

$$\mathrm{vec}([x]) = \frac{1}{N_1 N_2} (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1}) \mathrm{vec}([\hat{x}]) \tag{18}$$

From Equations (17) the Parseval's theorem can be proved directly:

$$\sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \left|\hat{x}_{k_1,k_2}\right|^2 = \mathrm{vec}([\hat{x}])^{\mathrm{H}} \mathrm{vec}([\hat{x}]) = \mathrm{vec}([x])^{\mathrm{H}} (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1}) (\overline{\mathbf{F}_{N_2}} \otimes \overline{\mathbf{F}_{N_1}}) \mathrm{vec}([x]) = \mathrm{vec}([x])^{\mathrm{H}} N_1 N_2 (\mathbf{I}_{N_2} \otimes \mathbf{I}_{N_1}) \mathrm{vec}([x]) = N_1 N_2 \mathrm{vec}([x])^{\mathrm{H}} \mathrm{vec}([x]) = N_1 N_2 \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} \left|x_{n_1,n_2}\right|^2.$$

The two-dimensional Fourier transform can also be written in a form of diagonalization:

$$\mathrm{diag}(\mathrm{vec}([\hat{x}])) = (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1})^{-1} \mathbf{X} (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1}) \tag{19}$$

$$\mathbf{X} = (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1}) \mathrm{diag}(\mathrm{vec}([\hat{x}])) (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1})^{-1} \tag{20}$$

Let x and h be two periodic sequences, $y = x \circledast h$, then

$$\mathbf{Y} = \mathbf{XH} = (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1}) \mathrm{diag}(\mathrm{vec}([\hat{x}])) (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1})^{-1} (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1}) \mathrm{diag}(\mathrm{vec}([\hat{h}])) (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1})^{-1} = (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1}) \mathrm{diag}(\mathrm{vec}([\hat{x}])) \mathrm{diag}(\mathrm{vec}([\hat{h}])) (\mathbf{F}_{N_2} \otimes \mathbf{F}_{N_1})^{-1}, \text{ so } \mathrm{diag}(\mathrm{vec}([\hat{y}])) = \mathrm{diag}(\mathrm{vec}([\hat{x}])) \mathrm{diag}(\mathrm{vec}([\hat{h}])).$$

This is just the convolution theorem.

6. Conclusion

The Fourier transform in digital image processing courses is presented from an angle of matrix algebra. Instead of the summation formulas typically used in courses on signals and systems and digital signal processing, matrix multiplication is employed to encapsulate additive and multiplicative operations, resulting in a concise representation. What is more, the Fourier transform is the eigenvalues of a circulant matrix and can be represented in a matrix diagonalization form. In this manner, students can leverage their knowledge from linear algebra courses, making it easier for them to grasp the fundamental concepts of Fourier transform in the digital image processing course.

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References

[1] Alan V. Oppenheim, Alan Willsky, and Barry Van Veen (1996). Systems and signals (2nd edition). Pearson.

[2] Simon Haykin and Barry Van Veen (2002). Systems and signals (2nd edition). John Wiley & Sons.

[3] Alan V. Oppenheim and Ronald W. Schafer (2010). Discrete-time signal processing (3rd edition). Pearson.

[4] John G. Proakis and Dimitris G. Manolakis (2022). Digital signal processing: principles, algorithms, and applications (5th edition). Pearson.

[5] Rafael C. Gonzalez and Richard E. Woods (2018). Digital image processing (4th edition). Pearson.

[6] Gilbert Strang (2022). Linear algebra and learning from data. Wellesley-Cambridge Press.