

An Asymptotic Formula for Estimating the Mean of A Class of Indices

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Abstract: Let $a, b, \beta \in \mathbf{R}^+$ with $b \leq a-1$, $\beta \geq 1$. This article uses the exponential pair principle and Vinogradov method to first obtain some bound exponential sums of the type $\sum_{N \leq n < N'} e(T^\beta / n^a)$ where $N < N' \leq 2N$, and then derives the asymptotic formula of $\sum_{mn^a \leq x^\beta} mn^b$. The asymptotic formula of $\sum_{mn^a \leq x^\beta} mn^b$ is an important tool for studying the mean estimation of the exponential divisor functions and divisor functions. By using the asymptotic formula of $\sum_{mn^a \leq x^\beta} mn^b$, we can derive more accurate mean estimates of the divisor function and exponential divisor function. Therefore, with the help of the asymptotic formula of $\sum_{mn^a \leq x^\beta} mn^b$, we can have a deeper understanding of the divisor function and exponential divisor function, and then provide solutions for more number theory problems.

Keywords: Asymptotic formula, Riemann Zeta function, The principle of index pairs, Vinogradov method.

1. Introduction

Let $a, b, \beta \in \mathbf{R}^+$ with $b \leq a-1$, $\beta \geq 1$, ζ denotes the Riemann zeta-function. Y.-F. S. Pétermann and J. Wu^[1] derived the following asymptotic formula using the exponential pair principle and Vinogradov method:

$$\sum_{mn^a \leq x} mn^b = \frac{1}{2} \zeta(2a-b)x^2 + \Delta(a, b; x), \quad (1)$$

Where

$$\Delta(a, b; x) \ll \begin{cases} x(\log x)^{\frac{2}{3}} & b = a-1, \\ x & b < a-1. \end{cases} \quad (2)$$

And the asymptotic formula was applied to obtain the asymptotic formula for the exponential divisor function. Some number theorists^[2-6] used the exponential pair principle and Vinogradov method to solve the mean problem of divisor functions and Euler functions. This article mainly extends $\sum_{mn^a \leq x} mn^b$ by extending $mn^a < x$ in $\sum_{mn^a \leq x} mn^b$ to $mn^a < x^\beta$, obtaining a more general asymptotic formula for $\sum_{mn^a \leq x^\beta} mn^b$. Similarly, based on Y.-F. S. Pétermann and J. Wu's research, the following results are obtained.

Theorem 1 Let $a, b, \beta \in \mathbf{R}^+$ with $b \leq a-1$, $\beta \geq 1$, ζ denotes the Riemann zeta-function. We have

$$\sum_{mn^a \leq x^\beta} mn^b = \frac{1}{2} \zeta(2a-b)x^{2\beta} + \Delta(a, b; x), \quad (3)$$

where

$$\Delta(a, b; x) \ll \begin{cases} x^\beta (\log x)^{2/3} & b = a-1, \\ x^\beta & b < a-1. \end{cases} \quad (4)$$

2. Preliminary Lemmas

Lemma 2.1 Let $a, b, \beta \in \mathbf{R}^+$ with $b \leq a-1$, $\beta \geq 1$, ζ denotes the Riemann zeta-function, we write $[t]$ for the integer part of t , $\{t\}$ for the decimal part of t . $\psi(t)$ for $\{t\} - \frac{1}{2}$. We have

$$\begin{aligned} \sum_{mn^a \leq x^\beta} mn^b &= \frac{1}{2} \zeta(2a-b)x^{2\beta} - x^\beta \sum_{n \leq x^{\beta/a}} \frac{1}{n^{a-b}} \psi\left(\frac{x^\beta}{n^a}\right) \\ &\quad + O\left(x^{\frac{\beta+\beta b}{a}}\right). \end{aligned} \quad (5)$$

Proof. Using the relation $[t] = t - \psi(t) - \frac{1}{2}$, we have

$$\begin{aligned} \sum_{mn^a \leq x^\beta} mn^b &= \sum_{n \leq x^{\frac{\beta}{a}}} n^b \sum_{m \leq \frac{x^\beta}{n^a}} m \\ &= \frac{1}{2} \sum_{n \leq x^{\frac{\beta}{a}}} n^b \left[\frac{x^\beta}{n^a} \right] \left(\left[\frac{x^\beta}{n^a} \right] + 1 \right) \\ &= \frac{1}{2} \sum_{n \leq x^{\frac{\beta}{a}}} n^b \left(\frac{x^\beta}{n^a} - \psi\left(\frac{x^\beta}{n^a}\right) - \frac{1}{2} \right) \left(\frac{x^\beta}{n^a} - \psi\left(\frac{x^\beta}{n^a}\right) + \frac{1}{2} \right) \\ &= \frac{1}{2} \zeta(2a-b)x^{2\beta} - x^\beta \sum_{n \leq x^{\frac{\beta}{a}}} \frac{1}{n^{a-b}} \psi\left(\frac{x^\beta}{n^a}\right) + O\left(x^{\frac{\beta+\beta b}{a}}\right). \end{aligned}$$

This completes the proof of lemma 2.1.

Lemma 2.2 Let $\beta \in \mathbf{R}^+$ with $\beta \geq 1$, $2 \leq z_1 < z_2 \leq x^\beta$ and $F(t) := (1/t)\psi(x^\beta/t^a)$, Denote by $V_F[z_1, z_2]$ the total

variation of F on $[z_1, z_2]$. Then

$$V_F[z_1, z_2] \ll \frac{x^\beta}{z_1^{a+1}} + \frac{1}{z_1}. \quad (6)$$

Proof. If $z_1 = t_0 < t_1 < \dots < t_n = z_2$ is a partition of the in $[z_1, z_2]$, then

$$\begin{aligned} & \sum_{k=1}^n |F(t_k) - F(t_{k-1})| \\ &= \sum_{k=1}^n \left| \frac{1}{t_k} \psi\left(\frac{x^\beta}{t_k^a}\right) - \frac{1}{t_{k-1}} \psi\left(\frac{x^\beta}{t_{k-1}^a}\right) \right| \\ &\leq \sum_{k=1}^n \left| \left(\frac{1}{t_k} - \frac{1}{t_{k-1}} \right) \psi\left(\frac{x^\beta}{t_k^a}\right) \right| + \sum_{k=1}^n \frac{1}{t_{k-1}} \left| \psi\left(\frac{x^\beta}{t_k^a}\right) - \psi\left(\frac{x^\beta}{t_{k-1}^a}\right) \right|. \end{aligned} \quad (7)$$

Since $\psi(t) \leq 1$ for all t , we have

$$\begin{aligned} & \sum_{k=1}^n \left| \left(\frac{1}{t_k} - \frac{1}{t_{k-1}} \right) \psi\left(\frac{x^\beta}{t_k^a}\right) \right| \leq \sum_{k=1}^n \left| \left(\frac{1}{t_k} - \frac{1}{t_{k-1}} \right) \right| \\ &\leq \frac{1}{t_0} - \frac{1}{t_n} < \frac{1}{z_1} - \frac{1}{z_2} < \frac{1}{z_1}. \end{aligned} \quad (8)$$

On the other hand, since $\psi(t)$ is of period 1, we have

$$\begin{aligned} & \sum_{k=1}^n \frac{1}{t_{k-1}} \left| \psi\left(\frac{x^\beta}{t_k^a}\right) - \psi\left(\frac{x^\beta}{t_{k-1}^a}\right) \right| \\ &\leq \frac{1}{z_1} \left(\frac{x^\beta}{z_1^{a+1}} + 1 \right) V_\psi[0, 1] \\ &\leq 2 \left(\frac{x^\beta}{z_1^{a+1}} + \frac{1}{z_1} \right). \end{aligned} \quad (9)$$

Substituting equations (8) and (9) into equation (7). This completes the proof of lemma 2.2.

Lemma 2.2 ^[1] Let M and P be integers, P being positive. $f(t)$ be a real function having a continuous $(k+1)th$ derivative, $k \geq 2$, in the interval $[M, M+P]$. Support that there exist positive absolute constants c_0, c_1, c_2, c_3 and c_4 such as $c_0 < 1$, $c_1 < 1$ and $c_2 + c_4 < c_1$. An integer r such that $c_0 k \leq r \leq k$; and number $s_j \geq 2$ ($j=1, \dots, r$) not exceeding k , such as for $M \leq t \leq M+P$ the following inequalities are satisfied:

- (i) $|f^{(k+1)}(t)/(k+1)!| \leq P^{-c_1(k+1)}$;
- (ii) $P^{-c_2 s_j} \leq |f^{(s_j)}(t)/s_j!| \leq P^{-c_3 s_j}$ ($j=1, \dots, r$).

Then for each positive integer P_1 not exceeding P , we have

$$\left| \sum_{M \leq m \leq M+P_1-1} e(f(m)) \right| \leq A P^{1-c/k^2}, \quad (10)$$

where $A > 0$ and $c > 0$ are absolute constants.

Lemma 2.4 Let $a > 0$, $e(t) = e^{2\pi i t}$, $\beta \in \mathbf{R}^+$ with $\beta \geq 1$,

$$e^{100(a+1)/(1-\{a\})} \max\{1, a^{-100/(1-\{a\})}\} \leq N < N' \leq 2N,$$

and let $T^\beta \geq N^{[a]+1}$. Then there exists a positive constant c_5 such that

$$\sum_{N \leq n < N'} e(T^\beta / n^a) \ll N \exp\{-c_5 \log^3 N / \log^2(T^\beta / N^a)\}. \quad (11)$$

Proof. We apply Lemma 2.3 to $f(t) := \frac{T^\beta}{t^a}$ with $M = N$,

$P = N$, $P_1 = N' - N$. For this, we choose

$$c_0 = \frac{1}{1000}, \quad c_1 = \frac{49 + \{a\}}{50}, \quad c_2 = \frac{77 + 23\{a\}}{100}, \quad c_3 = \frac{1}{4},$$

$$c_4 = \frac{1 - \{a\}}{100}, \quad k = \left\lceil \frac{100}{1 - \{a\}} \frac{\log(T^\beta / N^a)}{\log N} \right\rceil.$$

And take the s_j to be all integers s such that

$$\frac{2}{1 - \{a\}} \frac{\log(T^\beta / N^a)}{\log N} \leq s \leq \frac{4}{1 - \{a\}} \frac{\log(T^\beta / N^a)}{\log N}.$$

Obviously, the number r of s_j is between $c_0 k$ and k .

Next, we shall verify that $f(t) = \frac{T^\beta}{t^a}$ satisfies the conditions

(i) and (ii) of Lemma 2.3 with the parameters chosen above.

For condition (ii), when $N < t \leq 2N$, we have

$$\begin{aligned} \left| \frac{f^{(k+1)}(t)}{(k+1)!} \right| &= \prod_{\ell=1}^{k+1} \left(1 + \frac{a-1}{\ell} \right) T^\beta t^{-a-k-1} \\ &\leq e^{|a-1|(\log(k+1)+1)} T^\beta N^{-a-k-1} \\ &= N^{-\eta_1}, \end{aligned}$$

where

$$\begin{aligned} \eta_1 &:= k+1 - \frac{\log(T^\beta / N^a)}{\log N} - \frac{|a-1|(\log(k+1)+1)}{\log N} \\ &\geq \left(\frac{99 + \{a\}}{100} - \frac{(\log(k+1)+1)(1-\{a\})}{100(k+1)} \right) (k+1) \\ &\geq \frac{49 + \{a\}}{50} (k+1) = c_1(k+1), \end{aligned}$$

Then

$$\left| \frac{f^{(k+1)}(t)}{(k+1)!} \right| \leq N^{-c_1(k+1)} = P^{-c_1(k+1)}.$$

For condition (ii), when $N < t \leq 2N$, we first determine the lower bound of $|f^{(s_j)}(t)/s_j!|$,

$$\begin{aligned} |f^{(s_j)}(t)/s_j!| &= \prod_{\ell=1}^{s_j} \left(1 + \frac{a-1}{\ell} \right) T^\beta t^{-a-s_j} \\ &\geq \min\{a, 1\}^{s_j} T^\beta (2N)^{-a-s_j} \\ &= N^{-\eta_2}, \end{aligned}$$

where

$$\begin{aligned} \eta_2 &:= s_j - \frac{\log(T^\beta / N^a)}{\log N} - \frac{\log \min\{a, 1\}}{\log N} s_j + \frac{\log 2}{\log N} (s_j + a) \\ &\leq \frac{77 + 23\{a\}}{100} s_j \\ &= c_2 s_j, \end{aligned}$$

Then we determine the upper bound of $|f^{(s_j)}(t)/s_j!|$,

$$\begin{aligned}
\left| \frac{f^{(s_j)}(t)}{s_j!} \right| &= \prod_{\ell=1}^{s_j} \left(1 + \frac{a-1}{\ell} \right) T^\beta t^{-a-s_j} \\
&\leq e^{|a-1|(\log s_j + 1)} T^\beta N^{-a-s_j} \\
&\leq N^{-\eta_3},
\end{aligned}$$

where

$$\begin{aligned}
\eta_3 &:= s_j - \frac{\log(T^\beta / N^a)}{\log N} - \frac{|a-1|(\log s_j + 1)}{\log N} \\
&\geq \frac{1}{2} s_j - \frac{|a-1|(\log s_j + 1)}{100(a+1)} \\
&\geq \frac{1}{4} s_j = c_3 s_j.
\end{aligned}$$

Then we have

$$P^{-c_2 s_j} = N^{-c_2 s_j} \leq |f^{(s_j)}(t) / s_j!| \leq N^{-c_3 s_j} = P^{-c_3 s_j}.$$

According to Lemma 2.3, there exists two positive constants A and k such that

$$\begin{aligned}
\left| \sum_{N \leq n < N'} e(T^\beta / n^a) \right| &\leq AN^{1-c/k^2} \\
&\leq AN \exp\{-c_5 \log^3 N / \log^2(T^\beta / N^a)\},
\end{aligned}$$

where $c_5 := c(1 - \{a\})^2 / 10000$. This completes the proof of lemma 2.4.

Lemma 2.5 Let $\beta \in \mathbf{R}^+$ with $\beta \geq 1$, $a > 0$,

$$\exp\{c^*(\log x)^{2/3}\} \leq N \leq x^{\beta/([a]+1)}, \quad N < N' \leq 2N,$$

We have

$$\sum_{N \leq n < N'} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \ll e^{-c_6(\log N)^3 / (\log x)^2} (\log N)^3 / (\log x)^2, \quad (12)$$

$$\text{where } c_6 := \left(\beta + \frac{\beta^3}{([a]+1)^3} \right)^{-2} c_5, \quad c^* := (3c_6/5)^{-1/3},$$

$$c_5 := c(1 - \{a\})^2 / 10000.$$

Proof. By invoking a classical result on $\psi(t)$ we can write, for any $H \geq 1$,

$$\begin{aligned}
\sum_{N \leq n < N'} \psi(x / n^a) &\ll NH^{-1} + \sum_{1 \leq h \leq H} h^{-1} \left| \sum_{N \leq n < N'} e(hx / n^a) \right| \\
&\ll N \left(H^{-1} + e^{-c_5(\log N)^3 / \log^2(Hx^\beta / N^a)} \log H \right).
\end{aligned}$$

An application of Lemma 2.4 with $T^\beta = hx^\beta$ yields

$$\begin{aligned}
&\sum_{N \leq n < N'} \psi(x^\beta / n^a) \\
&\ll N \left(e^{-(\log N)^3 / (\log x)^2} + e^{-c_5(\log N)^3 / \log(x^{\beta^3/([a]+1)^3} \cdot x^\beta)^2} \log H \right) \\
&\ll N \left(H^{-1} + e^{-c_5(\log N)^3 / \log^2(Hx^\beta / N^a)} \log H \right).
\end{aligned}$$

Let $H = \exp\{(\log N)^3 / (\log x)^2\} \leq x^{\beta^3/([a]+1)^3}$, then

$$\begin{aligned}
&\sum_{N \leq n < N'} \psi(x^\beta / n^a) \\
&\ll N \left(e^{-(\log N)^3 / (\log x)^2} + e^{-c_6(\log N)^3 / (\log x)^2} (\log N)^3 / (\log x)^2 \right),
\end{aligned}$$

c_5 is small enough to ensure that the first equation can be absorbed by the second equation. Since

$$\exp\{c^*(\log x)^{2/3}\} \leq N \leq x^{\beta/([a]+1)},$$

we have

$$\sum_{N \leq n < N'} \psi(x^\beta / n^a) \ll N e^{-c_6(\log N)^3 / (\log x)^2} (\log N)^3 / (\log x)^2.$$

So

$$\begin{aligned}
\sum_{N \leq n < N'} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) &< \sum_{N \leq n < N'} \frac{1}{N} \psi\left(\frac{x^\beta}{n^a}\right) \\
&\ll e^{-c_6(\log N)^3 / (\log x)^2} (\log N)^3 / (\log x)^2.
\end{aligned}$$

This completes the proof of Lemma 2.5.

3. Proof of the Theorem 1

Proof. Let $\delta = \sum_{n \leq x^{\beta/a}} \frac{1}{n^{a-b}} \psi\left(\frac{x^\beta}{n^a}\right)$, according to the

Lemma 2.1, we have

$$\sum_{mn^a \leq x^\beta} mn^b = \frac{1}{2} \zeta(2a-b) x^{2\beta} - \delta x^\beta + O\left(x^{\frac{\beta+\beta b}{a}}\right).$$

Divide b in $\sum_{mn^a \leq x^\beta} mn^b$ into two cases: $b < a-1$ and

$b < a-1$.

When $b < a-1$, due to

$$\delta = \sum_{n \leq x^{\beta/a}} \frac{1}{n^{a-b}} \psi\left(\frac{x^\beta}{n^a}\right) \leq 1,$$

Therefore

$$\sum_{mn^a \leq x^\beta} mn^b = \frac{1}{2} \zeta(2a-b) x^{2\beta} + O(x^\beta). \quad (13)$$

When $b = a-1$, we divide $\delta = \sum_{n \leq x^{\beta/a}} \frac{1}{n^{a-b}} \psi\left(\frac{x^\beta}{n^a}\right)$ into

three cases:

$$\begin{aligned}
\delta &= \sum_{n \leq x^{\beta/a}} \frac{1}{n^{a-b}} \psi\left(\frac{x^\beta}{n^a}\right) \\
&= \sum_{n < w_0} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) + \sum_{w_0 \leq n \leq x^{\beta/([a]+1)}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) + \sum_{n < w_0} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \\
&= \delta_1 + \delta_2 + \delta_3,
\end{aligned}$$

where $w_0 = w_0(x) := \exp\{c^*(\log x)^{2/3}\}$. The following will prove one by one that

$$\delta_1 = \sum_{n < w_0} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \ll (\log x)^{2/3},$$

$$\delta_2 = \sum_{w_0 \leq n \leq x^{\beta/([a]+1)}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \ll (\log x)^{2/3},$$

and

$$\delta_3 = \sum_{n < w_0} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \ll 1.$$

Firstly, we prove that $\delta_1 \ll (\log x)^{2/3}$. According to $w_0 = \exp\{c^*(\log x)^{2/3}\}$, we have

$$\delta_1 = \sum_{n < w_0} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \ll \sum_{n < w_0} \frac{1}{n} \ll \log w_0 \ll (\log x)^{2/3}.$$

Next, we prove the inequality of $\delta_2 \ll (\log x)^{2/3}$. For

$[a] + 2 \leq j \leq J := [\frac{1}{c^*}(\log x)^{1/3}]$, we consider the sum

$$\sigma_j := \sum_{x^{\beta/j} \leq n < x^{\beta/(j-1)}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right).$$

Let $N_{\ell,j} := 2^\ell x^{\beta/j}$, L_j is a positive integer within the range of L values, L satisfies inequality $N_{L,j} < x^{\beta/(j-1)} \leq N_{L+1,j}$, Then we have

$$L_j \leq ((a+1)/a \log 2) j^{-2} \beta \log x, \\ j^{-1} \beta \log x \leq \log N_{\ell,j} \leq ((a+1)/a) j^{-1} \beta \log x.$$

For $j = [a] + 2, \dots, J$ and $\ell = 0, 1, \dots, L_j$. Using the Lemma 2.5 with

$$N = N_{\ell,j}, \quad N' = \begin{cases} N_{\ell+1,j} & \ell = 0, \dots, L_j - 1, \\ x^{1/(j-1)} & \ell = L_j, \end{cases}$$

then we have

$$\sigma_j \ll L_j \max_{0 \leq \ell \leq L_j} \left| \sum_{N_{\ell,j} \leq n < N_{\ell+1,j}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \right| \ll e^{-c_6 j^{-3} \log x} j^{-5} (\log x)^2.$$

We consider the function $\theta(t) := t^{-5} e^{-c_6 t^{-3} \log x}$. Since $\theta'(t) \geq 0$ holds when $0 < t \leq J$, $\theta(t)$ is an increasing function in the interval $(0, J]$. Then

$$\sigma_j \ll e^{-c_6 j^{-3} \log x} J^{-5} (\log x)^2.$$

In addition, we find that

$$\begin{aligned} \sum_{w_0 < n \leq x^{1/J}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) &\ll \sum_{w_0 < n \leq x^{1/J}} \frac{1}{n} \\ &\ll \log(x^{1/J} / w_0) + 1 / w_0 \\ &\ll (\log x)^{1/3}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \delta_2 &= \sum_{w_0 \leq n \leq x^{\beta/([a]+1)}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \\ &= \sum_{[a]+2 \leq j \leq J} \sigma_j + O((\log x)^{1/3}) \\ &\ll (\log x)^{2/3}. \end{aligned}$$

Finally, we prove that $\delta_3 \ll 1$. Let $F(t)$ be a function of bounded variation on $[n, n+1]$ for each integer n , then we write

$$\sum_{z_1 < n \leq z_2} F(n) = \int_{z_1}^{z_2} F(t) dt + \frac{1}{2} (F(z_1) + F(z_2)) + O(V_F[z_1, z_2]).$$

Let $F(t) = \psi(x^\beta / t^a) / t$. Using Lemma 2.2 with

$z_1 = [z(x)]$, $z_2 = [x^{\beta/a}]$ and $u = \frac{x^\beta}{t^a}$ we obtain

$$\sum_{z < n \leq x^{1/a}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) = \frac{1}{a} \int_1^\infty \frac{\psi(u)}{u} du + O\left(\frac{x^\beta}{z_1^{a+1}} + \frac{1}{z_1} + \frac{z_1^a}{x^\beta}\right).$$

Putting $z_1 = x^{\frac{\beta}{[a]+1}}$, we have

$$\delta_3 = \sum_{x^{\beta/([a]+1)} < n \leq x^{\beta/a}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \ll 1.$$

Then

$$\delta = \sum_{n \leq x^{\beta/a}} \frac{1}{n} \psi\left(\frac{x^\beta}{n^a}\right) \ll (\log x)^{2/3}.$$

When $b = a - 1$, we obtain

$$\begin{aligned} \sum_{mn^a \leq x^\beta} mn^b &= \frac{1}{2} \zeta(2a-b) x^{2\beta} - x^\beta \sum_{n \leq x^{\beta/a}} \frac{1}{n^{a-b}} \psi\left(\frac{x^\beta}{n^a}\right) + O\left(x^{\frac{\beta+\beta b}{a}}\right) \\ &= \frac{1}{2} \zeta(2a-b) x^{2\beta} + O((\log x)^{2/3}). \end{aligned}$$

This completes the proof of Theorem 1.

4. Conclusion

The asymptotic formula of $\sum_{mn^a \leq x^\beta} mn^b$ plays an important role in studying exponential divisor functions and divisor functions. This article mainly uses the exponential pair principle and Vinogradov method to obtain the asymptotic formula of $\sum_{mn^a \leq x^\beta} mn^b$, further extending $\sum_{mn^a \leq x} mn^b$. On this basis, we can deeper study the mean estimation of the divisor functions and exponential divisor functions.

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