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Diophantine Approximation with Prime Variables and Mixed Powers

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Abstract: Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be non-zero real numbers, not all negative, and let λ_1/λ_2 be irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence and $\delta > 0$ .We prove that for any given $\varepsilon > 0$,the number of v satisfying $v \in \mathcal{V}$ and $v \le X$ for which $\left|\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^4 - v\right| < v^{-\delta}$ has no solution in primes p_1, p_2, p_3, p_4, p_5 does not exceed $O\left(X^{17/18+2\delta+\varepsilon}\right)$.

Keywords: Diophantine inequalities; Davenport-Heilbronn method; prime.

1. Introduction

In the field of number theory, Diophantine approximation and prime number theory are two important research directions, and many analytic number theorists have established in-depth research on them and achieved corresponding results. In 1996, Brüdern[1] et al first studied the exception set problem of prime variable bilinear type. Later, Cai Yingchun[2] and Wang Yuchao [3] made further improvements to this. In 2001, Cook [4] et al. studied the problem of the exception set of quadratic prime variables Diophantine inequalities. Later, Harman [5] further improved this. In recent years, Cook[6], Li Weiping and Wang Tianze[7], Li Weiping[8] et al. and Mou Quanwu [9] and other studies have also proved some related problems related to the mixed powers of prime variables.

For the sake of research, a definition of a good interval is first introduced. If there are normal numbers $c > 0$ such that $\mathcal{V}$ any two unequal positive real numbers in u the pair, v , there are $|u-v| > c$, and the increasing sequence of positive real numbers $\mathcal{V}$ is called a well-spaced sequence.

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ be non-zero real numbers that are not all negative, λ_1/λ_2 and are irrational and algebraic numbers $\mathcal{V}$ is a well-spaced sequence that $\delta > 0$ denotes $\mathbb{E}(\mathcal{V}, N, \delta)$ the set of and so $\lambda_1 p_1^3 + \lambda_2 p_2^3 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^4 - v\Big| < v^{-\delta}$ that the inequality of prime variables has no prime solution p_1 , p_2 , p_3 , p_4 , p_5 , $v \in \mathcal{V}$ and $v \le N$. For any $\varepsilon > 0$, let $E(\mathcal{V}, N, \delta) = \left|\mathbb{E}(\mathcal{V}, X, \delta)\right|$ .In this paper, some conclusions of [5] and [10] are mainly used to obtain the following theorems:

Theorem 1.1 Let λ_1 , λ_2 , λ_3 , λ_4 , λ_5 be non-zero real numbers that are not all negative, λ_1/λ_2 and be irrational and algebraic numbers $\mathcal{V}$ is a sequence with good spacing, $\delta > 0$ then, for $\varepsilon > 0$, there is

$$E(\mathcal{V}, N, \delta) \ll X^{1+2\delta-1/18+\varepsilon}. \tag{1.1}$$

Theorem 1.2 Let λ_1 , λ_2 , λ_3 , λ_4 , λ_5 be non-zero real numbers that are not all negative λ_5, and λ_1/λ_2 are irrational and algebraic numbers. $\mathcal{V}$ is a well-spaced sequence, $\delta > 0$ and for any $\varepsilon > 0$, there is a sequence $N_j \to \infty$ satisfied

$$E(\mathcal{V}, N, \delta) \ll X_j^{1+2\delta-1/18+\varepsilon}. \tag{1.2}$$

Further, if an irrational number λ_1/λ_2 is rationally approximated to the denominator of the fractional sequence q_j, it is satisfied

$$q_{j+1}^{1-\omega} \ll q_j, \tag{1.3}$$

Where $\omega \in [0,1)$, then, to any $N \ge 1$ and $\varepsilon > 0$, there is

$$E(\mathcal{V}, N, \delta) \ll X^{1+2\delta-\chi+\varepsilon}. \tag{1.4}$$

Where

$$\chi = \min\left(\frac{1-\omega}{3-2\omega}, \frac{1}{18}\right). \tag{1.5}$$

It is known from the famous Roth's theorem that when λ_1/λ_2 are irrational numbers and algebraic numbers, the above ω can take any sufficiently small positive number, then theorem 1.1 is easily obtained by theorem 1.2. Below it is only necessary to prove theorem 1.2.

2. Overview of Symbols and Methods

In this article, letters p with subscripts and subscripts both represent prime numbers, ε and δ represent a fixed arbitrarily small normal number η is a sufficiently small fixed integer , $0<\tau<1$, O , $\ll$and $\gg$the implicit constant in the sign usually depends at most on $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\varepsilon$. δ . In the text $X=q^{9/8}$ and $\tau=X^{-\delta}$, $e(\alpha)=e^{2\pi i\alpha}$.

Suppose N is a sufficiently large number, let $\mathbb{E}(\mathcal{J})=\mathbb{E}(\mathcal{V},N,\delta)\cap\mathcal{J}$, $E(\mathcal{J})=|\mathbb{E}(\mathcal{J})|$, where $\mathcal{J}$ is a $[0,N]$ subset of . It is known by $\mathcal{V}$ the definition of $\mathbb{E}(\mathcal{V},N,\delta)$ a good interval set and sum

$$E(\mathcal{V},N,\delta)=E\left(\left[N^{17/18},N\right]\right)+E\left(\left[0,N^{17/18}\right]\right) \ll E\left(\left[N^{17/18},N\right]\right)+N^{17/18}.$$

Therefore it is only necessary to prove $E\left(\left[N^{17/18},N\right]\right)$ the estimation on the basis of the dichotomy method. Set $N^{17/18}\le X\le N$ will be estimated $E\left(\left[\frac{X}{2},X\right]\right)$ for those v who meet $\frac{X}{2}\le v\le X$ Using the Davenport-Heilbronn method (see Davenport and Heilbronn [11]).

Let $\tau>0,\alpha\neq 0$

$$K_\tau(\alpha)=\left(\frac{\sin(\pi\tau\alpha)}{\pi\alpha}\right)^2.$$

By text [1], there is

$$K_\tau(\alpha)\ll\min\left(\tau^2,|\alpha|^{-2}\right), \tag{2.1}$$

$$A(y)=\int_{-\infty}^{+\infty}e(\alpha y)K_\tau(\alpha)\,dy=\max(0,\tau-|y|), \tag{2.2}$$

Definition $k=3,4$, there is

$$S_k(\alpha)=(\log p)e\left(\alpha p^k\right), U_k(\alpha)=\sum_{n\in\mathcal{I}_k}e\left(\alpha n^k\right),$$
$$\mathcal{I}_k=\left[(\eta X)^{1/k},X^{1/k}\right]. \tag{2.3}$$

For the convenience $\mathbb{R}$ of the measurable subset of the denoted $\aleph$ as:

$$I(v,X;\aleph)=\int_\aleph S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)\,d\alpha$$

And then for $i=1,2,3,\ f=4,5,\ j=1,2,3,4,5,$ write

$$I(v,X;\mathbb{R})=\int_\mathbb{R} S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha) S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)\,d\alpha$$
$$=\sum_{\substack{p_i\in I_3\\ p_f\in I_4}}\prod_{j=1}^5(\log p_j)\int_{-\infty}^{+\infty}e\left(\left(\lambda_1p_1^3+\lambda_2p_2^3+\lambda_3p_3^3+\lambda_4p_4^4+\lambda_5p_5^4-v\right)\alpha\right)K_\tau(\alpha)\,d\alpha$$
$$=\sum_{\substack{p_i\in I_3\\ p_f\in I_4}}\prod_{j=1}^5(\log p_j)\max\left(0,\left|\lambda_1p_1^3+\lambda_2p_2^3+\lambda_3p_3^3+\lambda_4p_4^4+\lambda_5p_5^4-v\right|\right)$$
$$\ll\tau\wp(v,X)(\log X)^5. \tag{2.4}$$

where $\wp(v,X)$ inequality is described

$$\left|\lambda_1p_1^3+\lambda_2p_2^3+\lambda_3p_3^3+\lambda_4p_4^4+\lambda_5p_5^4-v\right|<v^{-\delta}$$

the number of prime p_1,p_2,p_3,p_4,p_5 solutions v .

To estimate the integral of equation (2.4), divide the real interval into three parts:

$$\mathcal{M}=\{\alpha:|\alpha|\le\vartheta\}, c=\{\alpha:\vartheta<|\alpha|\le\varsigma\},$$
$$r=\{\alpha:|\alpha|>\varsigma\},$$

They are called the main interval, the residual interval, and the ordinary interval. Where $\vartheta=X^{-7/9-\varepsilon}$, $\varsigma=\tau^{-2}X^{3/4+2\varepsilon}$.

3. Prepare the Lemma

Lemma 3.1[11] Let be $k\ge 1$ a real number, and for any fixed real number $A\ge 6$, there is

$$\int_{|\alpha|\le X^{-1+(5/6k)-\varepsilon}}\left|S_k(\alpha)-U_k(\alpha)\right|^2d\alpha\ll X^{2/k-1}(\log X)^{-A}.$$

Lemma 3.2[12] If $X^{1/3}\ge Y\ge X^{1/3-1/36+\varepsilon}$, $|S_3(\lambda_3\alpha)|>Y$ There are two coprime integers that a , q satisfy

$$1\le q\ll\left(\frac{X^{1/3+\varepsilon}}{Y}\right)^2,\ |q\lambda\alpha-a|\ll X^{-1}\left(\frac{X^{1/3+\varepsilon}}{Y}\right)^2.$$

Lemma 3.3[13] Let be α a real number, exist $a\in\mathbb{Z}$, $q\in\mathbb{N}$ satisfy

$$(a,q)=1, |q\alpha-a|<q^{-1} \tag{3.1}$$

For any real number $\varepsilon>0$, a positive integer $k\ge 2$, there is

$$\sum_{1\le p\le\mathbb{N}}(\log p)e\left(p^k\alpha\right)\ll N^{1+\varepsilon}\left(\frac{1}{q}+N^{-1/2}+\frac{q}{N^k}\right)^{4^{1-k}}.$$

Corollary 3.4 Let $X^{-7/9}\ge|\alpha|\ge X^{-19/24-\varepsilon}$

$$|S_3(\lambda\alpha)|\ll X^{1/3-1/96+\varepsilon}, j=1,2,3,$$

Proof In equation (3.1), $q=\left[|\lambda\alpha|^{-1}\right]$,take $a=1$, apparently satisfies $|q\lambda\alpha-a|<q^{-1}$ .At this time $X^{7/9}\gg q\gg X^{19/24+\varepsilon}$, it is known from the definition of $S_3(\lambda\alpha)$ lemma 3.3 and that the $|S_3(\lambda\alpha)|\ll X^{1/3}\left(q^{-1/16}+X^{-1/96}+q^{1/16}+X^{-1/16}\right)\ll X^{1/3-1/96+\varepsilon}$. proof is complete.

Lemma 3.5[10] For any positive integer m , $1\le m\le k$,

$2 \le k \le 4$ there is

$$\int_{-1}^{1}\left|S_k(\alpha)\right|^{2^m} d\alpha \ll X^{\frac{2^m-m}{k}+\varepsilon},$$

$$\int_{-\infty}^{\infty}\left|S_k(\alpha)\right|^{2^m} K_\tau(\alpha)\,d\alpha \ll \tau X^{\frac{2^m-m}{k}+\varepsilon}.$$

Lemma 3.6[14] For any $\varepsilon > 0$, there is

$$\int_{-\infty}^{\infty}\left|S_3(\lambda\alpha)\right|^2\left|S_4(\lambda\alpha)\right|^4 K_\tau(\alpha)\,d\alpha \ll \tau X^{1/6+\varepsilon}.$$

4. The Main Arc

In this part, we calculate the integrals on the main interval, and now further divide the main interval $\mathcal{M}$ into two sub-intervals $\mathcal{M}_1$ and $\mathcal{M}_2$ define them separately

$$\mathcal{M}_1 = \left\{\alpha : |\alpha| \le X^{5/24-1-\varepsilon}\right\},$$

$$\mathcal{M}_2 = \left\{\alpha : X^{5/24-1-\varepsilon} < |\alpha| \le X^{-7/9-\varepsilon}\right\}.$$

For $k = 3,4$, set $T_k(\alpha) = \int_{\mathcal{I}_k} e\left(\alpha t^k\right)dt$, there is

$$T_i(\alpha) \ll X^{1/i-1}\min\left(X, |\alpha|^{-1}\right). \tag{4.1}$$

4.1. The region $\mathcal{M}_1$

In this subsection, we will give the $\mathcal{M}_1$ lower bound of the integral on the interval

Lemma 4.1. We have

$$I(v, X; \mathcal{M}_1) \gg \tau^2 X^{1/2}. \tag{4.2}$$

Proof. It is easy to show that

$$\begin{aligned}
&I(v, X; \mathcal{M}_1)\\
&\le \int_{\mathcal{M}_1} T_3(\lambda_1\alpha)T_3(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)\\
&T_4(\lambda_5\alpha)e(-v\alpha)K_\tau(\alpha)\,d\alpha\\
&+\int_{\mathcal{M}_1}\left(S_3(\lambda_1\alpha)-T_3(\lambda_1\alpha)\right)T_3(\lambda_2\alpha)T_3(\lambda_3\alpha)\\
&T_4(\lambda_4\alpha)T_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)\,d\alpha\\
&+\int_{\mathcal{M}_1} S_3(\lambda_1\alpha)\left(S_3(\lambda_2\alpha)-T_3(\lambda_2\alpha)\right)T_3(\lambda_3\alpha)\\
&T_4(\lambda_4\alpha)T_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)\,d\alpha\\
&+\int_{\mathcal{M}_1} S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)\left(S_3(\lambda_3\alpha)-T_3(\lambda_3\alpha)\right)\\
&T_4(\lambda_4\alpha)T_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)\,d\alpha\\
&+\int_{\mathcal{M}_1} S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)\left(S_4(\lambda_4\alpha)-\right.\\
&\left.T_4(\lambda_4\alpha)\right)T_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)\,d\alpha\\
&+\int_{\mathcal{M}_1} S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)\\
&\left(S_4(\lambda_5\alpha)-T_4(\lambda_5\alpha)\right)K_\tau(\alpha)e(-v\alpha)\,d\alpha\\
&:= J_0 + J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

The following will prove one by one $J_0 \gg \tau^2 X^{1/2}$, and $J_j = o\left(\tau^2 X^{1/2}\right), 1 \le j \le 5$ so that the lemma can be proven.

First, determine J_0 the lower definite boundary

$$\begin{aligned}
&J_0 = \int_{\mathbb{R}} T_3(\lambda_1\alpha)T_3(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_4(\lambda_5\alpha)\\
&e(-v\alpha)K_\tau(\alpha)\,d\alpha\\
&+O\Big(\int_{|\alpha|>X^{5/24-1-\varepsilon}}\big|T_3(\lambda_1\alpha)T_3(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)\\
&T_4(\lambda_5\alpha)e(-v\alpha)K_\tau(\alpha)\big|\,d\alpha\Big).
\end{aligned} \tag{4.3}$$

Consists of (2.1) and (4.). 1) Determine J_0 the error term

$$\begin{aligned}
&\ll \tau^2 X^{-7/2}\int_{|\alpha|>X^{5/24-1-\varepsilon}}|\alpha|^{-5}\,d\alpha\\
&\ll \tau^2 X^{-7/2+19/6+4\varepsilon} = o\left(\tau^2 X^{1/2}\right).
\end{aligned} \tag{4.4}$$

For convenience, not $\mathcal{Y} = (y_1, y_2, y_3, y_4, y_5)$ by replacing the integral variables, order

$$y_i = \lambda_i p_i^3, i = 1,2,3,\ y_k = \lambda_k p_k^4, k = 4,5,$$

The $\mathcal{Y} = (y_1, y_2, y_3, y_4, y_5)$ interval where is $\left[\lambda_j\eta X, \lambda_j X\right]$ and takes the integral satisfies $\left|\sum_{j=1}^{5} y_j - v\right| < \tau$,and takes $\left|\sum_{j=1}^{5} y_j - v\right| < \frac{\tau}{2}$, there is $-\sum_{j=1}^{5} y_j + v - \frac{\tau}{2} < y_5 < -\sum_{j=1}^{5} y_j + v$, then

$$\begin{aligned}
&\int_{\mathbb{R}} T_3(\lambda_1\alpha)T_3(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_4(\lambda_5\alpha)\\
&e(-v\alpha)K_\tau(\alpha)\,d\alpha\\
&=\frac{1}{532(\lambda_1\lambda_2\lambda_3)^{2/3}(\lambda_4\lambda_5)^{3/4}}\int (y_1y_2y_3)^{-2/3}(y_4y_5)^{-3/4}\\
&\int_{\mathbb{R}} e\left(\left(\sum_{j=1}^{5} y_j - v\right)\alpha\right)K_\tau(\alpha)\,d\alpha\,dy\\
&\gg \int (y_1y_2y_3)^{-2/3}(y_4y_5)^{-3/4}\int_{\mathbb{R}}\max\left(0, \left|\sum_{j=1}^{5} y_j - v\right|\right)d\mathcal{Y}\\
&\gg \tau X^{-3/4}\int_{\eta\lambda_1 X}^{\lambda_1 X}\cdots\int_{\eta\lambda_4 X}^{\lambda_4 X}(y_1y_2y_3)^{-2/3}(y_4y_5)^{-3/4}\,dy_1\cdots dy_4\\
&\gg \tau^2 X^{1/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{\mathbb{R}} T_3(\lambda_1\alpha)T_3(\lambda_2\alpha)T_3(\lambda_3\alpha)T_4(\lambda_4\alpha)T_4(\lambda_5\alpha)\\
&e(-v\alpha)K_\tau(\alpha)\,d\alpha \gg \tau^2 X^{1/2}.
\end{aligned} \tag{4.5}$$

Determined by the formula (4.3), (4.4) and (4.5),

$$J_0 \gg \tau^2 X^{1/2}. \tag{4.6}$$

The next processing J_1, summed by Euler, has it

$$\left|U_k(\alpha)-T_k(\alpha)\right| \ll 1+|\alpha| X \text{ .where } k=3,4. \quad (4.7)$$

There is by (2.1).

$$\begin{aligned} J_1 &\ll \tau^2 \int_{\mathcal{M}_1} \left(S_3(\lambda_1\alpha)-T_3(\lambda_1\alpha)\right) T_3(\lambda_2\alpha) T_3(\lambda_3\alpha) \\ &\quad T_4(\lambda_4\alpha) T_4(\lambda_5\alpha) d\alpha \\ &\ll \tau^2 \int_{\mathcal{M}_1} \left(S_3(\lambda_1\alpha)-U_3(\lambda_1\alpha)\right) T_3(\lambda_2\alpha) T_3(\lambda_3\alpha) \\ &\quad T_4(\lambda_4\alpha) T_4(\lambda_5\alpha) d\alpha \\ &+\tau^2 \int_{\mathcal{M}_1} \left(U_3(\lambda_1\alpha)-T_3(\lambda_1\alpha)\right) T_3(\lambda_2\alpha) T_3(\lambda_3\alpha) \\ &\quad T_4(\lambda_4\alpha) T_4(\lambda_5\alpha) d\alpha \\ &:= \tau^2 (A_1+B_1) \end{aligned} \quad (4.8)$$

By Cauchy's inequality, lemma 3.1 and (4.1),we have

$$\begin{aligned} A_1 &\ll X^{1/3+1/4+1/4} \left(\int_{\mathcal{M}_1} \left|S_3(\lambda_1\alpha)-U_3(\lambda_1\alpha)\right|^2 d\alpha\right)^{1/2} \\ &\quad \left(\int_{\mathcal{M}_1} \left|T_3(\lambda_2\alpha)\right|^2 d\alpha\right)^{1/2} \\ &\ll X^{5/6} (\log X)^{-A/2} \left(\int_0^{1/X} X^{2/3} d\alpha + \right. \\ &\quad \left. \int_{1/X}^{X^{-19/24-\varepsilon}} X^{-4/3} |\alpha|^{-2} d\alpha\right)^{1/2} \\ &\ll X^{1/2} (\log X)^{-A/2}. \end{aligned} \quad (4.9)$$

Equations (4.1) and (4.7),we have

$$\begin{aligned} B_1 &\ll \int_0^{1/X} \left|T_3(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_4(\lambda_5\alpha)\right| d\alpha \\ &+X \int_{1/X}^{X^{-19/24-\varepsilon}} |\alpha| \left|T_3(\lambda_2\alpha) T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_4(\lambda_5\alpha)\right| d\alpha \\ &\ll X^{1/6} + X \int_{1/X}^{X^{-19/24-\varepsilon}} |\alpha|^{-3} X^{-17/6} d\alpha \ll X^{1/6}. \end{aligned} \quad (4.10)$$

Determined by the formula (47)- (4.9),we have

$$J_1 = o\left(\tau^2 X^{1/2}\right) \quad (4.11)$$

Next calculated J_2, by (2.1) there is

$$\begin{aligned} J_2 &\ll \tau^2 \int_{\mathcal{M}_1} S_3(\lambda_1\alpha) \left(S_3(\lambda_2\alpha)-T_3(\lambda_2\alpha)\right) T_3(\lambda_3\alpha) \\ &\quad T_4(\lambda_4\alpha) T_4(\lambda_5\alpha) d\alpha \\ &\ll \tau^2 \int_{\mathcal{M}_1} S_3(\lambda_1\alpha) \left(S_3(\lambda_2\alpha)-U_3(\lambda_2\alpha)\right) T_3(\lambda_3\alpha) \\ &\quad T_4(\lambda_4\alpha) T_4(\lambda_5\alpha) d\alpha \\ &+\tau^2 \int_{\mathcal{M}_1} S_3(\lambda_1\alpha) \left(U_3(\lambda_2\alpha)-T_3(\lambda_2\alpha)\right) T_3(\lambda_3\alpha) \\ &\quad T_4(\lambda_4\alpha) T_4(\lambda_5\alpha) d\alpha \\ &:= \tau^2 (A_2+B_2). \end{aligned} \quad (4.12)$$

According to Cauchy's inequality, Lemma 3.1 is known

$$\begin{aligned} A_2 &\ll X^{5/6} \left(\int_{\mathcal{M}_1} \left|S_3(\lambda_1\alpha)\right|^2 d\alpha\right)^{1/2} \left(\int_{\mathcal{M}_1} \left|S_3(\lambda_2\alpha)- \right.\right. \\ &\quad \left.\left. U_3(\lambda_2\alpha)\right|^2 d\alpha\right)^{1/2} \\ &\ll X^{5/6} (\log X)^{-A/2} X^{-1/6-1/6} \\ &\ll X^{1/2} (\log X)^{-A/2}. \end{aligned} \quad (4.13)$$

For B_2 this, it is determined by the equations (4.1) and (4.7) that

$$\begin{aligned} B_2 &\ll \int_0^{1/X} \left|S_3(\lambda_1\alpha)\right| \left|T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_4(\lambda_5\alpha)\right| d\alpha \\ &+X \int_{1/X}^{X^{-19/24-\varepsilon}} |\alpha| \left|S_3(\lambda_1\alpha)\right| \left|T_3(\lambda_3\alpha) T_4(\lambda_4\alpha) T_4(\lambda_5\alpha)\right| d\alpha \\ &\ll X^{1/6} + X^{-13/12} \int_{1/X}^{X^{-19/24-\varepsilon}} |\alpha|^{-2} \left|S_3(\lambda_1\alpha)\right| d\alpha \\ &\ll X^{1/6}. \end{aligned} \quad (4.14)$$

Determined by the formula (4.12)- (4.14),we obtain

$$J_2 = o\left(\tau^2 X^{1/2}\right) \quad (4.15)$$

In the J_2 same way, with the help of Cauchy's inequality, lemma 3.1, there is

$$J_s = o\left(\tau^2 X^{1/2}\right), s = 3,4,5. \quad (4.16)$$

Simultaneous formula (4.6), (4.11), (4.14) and (4.16), the lemma is prove.

4.2. The region $\mathcal{M}_2$

Lemma 4.2. We have

$$I(v, X; \mathcal{M}_2) \gg \tau^2 X^{1/2}. \quad (4.17)$$

Proof. By lemma 3.4, 3.5 and 3.6 we have

$$\begin{aligned} I(v, X; \mathcal{M}_2) &\ll \tau^2 X^{1/3-1/96+\varepsilon} \left(\int_{\mathcal{M}_2} \left|S_3(\lambda_2\alpha)\right|^2 d\alpha\right)^{1/2} \\ &\times \left(\int_{\mathcal{M}_2} \left|S_3(\lambda_2\alpha)\right|^2 \left|S_4(\lambda_4\alpha)\right|^4 d\alpha\right)^{1/4} \\ &\times \left(\int_{\mathcal{M}_2} \left|S_3(\lambda_3\alpha)\right|^2 \left|S_4(\lambda_5\alpha)\right|^4 d\alpha\right)^{1/4} \end{aligned}$$

$$\ll \tau^2 X^{1/3-1/96+\varepsilon} X^{1/6} \ll \tau^2 X^{1/2-1/96+\varepsilon}$$
$$= o\left(\tau^2 X^{1/2}\right).$$

Then by lemma 4.1 and 4.2 the integral over the interval is obtained $\mathcal{M}$, i.e., the following lemma:

Lemma 4.3. We have
$$I(v, X; \mathcal{M}) \gg \tau^2 X^{1/2}.$$

5. The Trivial Arc

In this section, the integral estimation of r the trivial interval will be done, there are

Lemma 5.1. We have
$$I(v, X; r) \ll \tau^2 X^{1/2-\varepsilon}.$$

Proof. By Cauchy's inequality and $S_3(\lambda_i\alpha)$ the trivial estimation $i = 1, 2, 3$,

$$I(v, X; r) \ll X^{1/3+1/3+1/3}\left(\int_r \left|S_4(\lambda_4\alpha)\right|^2 K_\tau(\alpha)\, d\alpha\right)^{1/2}$$
$$\left(\int_r \left\|S_4(\lambda_5\alpha)\right|^2 K_\tau(\alpha)\, d\alpha\right|\right)^{1/2}. \tag{5.1}$$

It is obtained by the periodicity of (2.1) and Lemma 3.5

$$\int_r \left|S_4(\lambda_4\alpha)\right|^2 K_\tau(\alpha)\, d\alpha \ll \int_{|\alpha|\geq\varsigma} \left|S_4(\lambda_4\alpha)\right|^2 |\alpha|^{-2}\, d\alpha$$
$$\ll \int_{|\alpha|\geq|\lambda_4|\varsigma} \left|S_4(\lambda_4\alpha)\right|^2 |\alpha|^{-2}\, d\alpha$$
$$\ll \sum_{m=[|\lambda_4|\varsigma]}^{\infty} \int_m^{m+1} \left|S_4(\lambda_4\alpha)\right|^2 |\alpha|^{-2}\, d\alpha$$
$$\ll \sum_{m=[|\lambda_4|\varsigma]}^{\infty} \int_m^{m+1} \left|S_4(\lambda_4\alpha)\right|^2 m^{-2}\, d\alpha$$
$$\ll \int_0^1 \left|S_4(\alpha)\right|^2 d\alpha \sum_{m=[|\lambda_4|\varsigma]}^{\infty} \int_m^{m+1} m^{-2}$$
$$\ll X^{1/4+\varepsilon}\varsigma^{-1}. \tag{5.2}$$

The same goes for

$$\int_r \left|S_4(\lambda_4\alpha)\right|^2 K_\tau(\alpha)\, d\alpha \ll X^{1/4+\varepsilon}\varsigma^{-1}. \tag{5.3}$$

Joint formula (5.1) - (5.3) ,we have

$$I(v, X; r) \ll X^{1+1/4+\varepsilon} \ll \tau^2 X^{1/2-\varepsilon}. \tag{5.4}$$

6. The Minor Arc

In this section, the integrals on the estimated remainder interval c are obtained, for a fixed parameter N is a a/q rational approximation fraction for an λ_1/λ_2 irrational number, and

$$N = q^{(18/17)\cdot(9/8)}. \tag{6.1}$$

Set $c' = c_1 \setminus c_2$, $\hat{c} = c \setminus c'$,we have
$$c_i = \left\{\alpha \in c, \left|S_3(\lambda_i\alpha)\right| < X^{1/3-(1/2)\sigma+2\varepsilon}\right\},$$
$i = 1, 2.$

Where
$$\sigma = \frac{1}{18}.$$

Lemma 6.1. We have

$$\int_{c'} \left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)\right.$$
$$\left.S_4(\lambda_5\alpha)\right|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{2-\sigma+\varepsilon}. \tag{6.2}$$

Proof. For c' the definition, c_1 and c_2 similar to the situation, only the integral is given c_1 here, as determined by the Cauchy-Schwarz inequality, and the lemma 3.5 there

$$\int_{c_1} \left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)\right|^2$$
$$K_\tau(\alpha)\, d\alpha$$
$$\ll \max_{\alpha\in c_1}\left|S_3(\lambda_1\alpha)\right|^2 \left(\int_{c_1}\left|S_3(\lambda_2\alpha)\right|^4 K_\tau(\alpha)\, d\alpha\right)^{1/2}$$
$$\times\left(\int_{c_1}\left|S_3(\lambda_3\alpha)\right|^8 K_\tau(\alpha)\, d\alpha\right)^{1/4}$$
$$\times\left(\int_{c_1}\left|S_4(\lambda_4\alpha)\right|^{16} K_\tau(\alpha)\, d\alpha\right)^{1/8}$$
$$\times\left(\int_{c_1}\left|S_4(\lambda_5\alpha)\right|^{16} K_\tau(\alpha)\, d\alpha\right)^{1/8}$$
$$\ll \tau X^{2-\sigma+\varepsilon}.$$

Lemma 6.12. We have

$$\int_{\hat{c}} \left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)\right.$$
$$\left.S_4(\lambda_5\alpha)\right|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{2-16\sigma+\varepsilon}. \tag{6.3}$$

Proof. Using Harman's method in [5], the intervals $\hat{c}$ are divided into disjoint sets $A(Y_1, Y_2, u)$

$$A(Y_1, Y_2, u) = \left\{\alpha \in \hat{c} : Y_1 \leq \left|S_3(\lambda_1\alpha)\right| < 2Y_1,\right\}.$$
$$Y_2 \leq \left|S_3(\lambda_2\alpha)\right| < 2Y_2; u \leq |\alpha| < 2u\}.$$

Here $Y_1 = X^{1/3-(1/2)\sigma+2\varepsilon}2^{t_1}$, $Y_2 = X^{1/3-(1/2)\sigma+2\varepsilon}2^{t_2}$, $u = \varsigma 2^{t_3}$, t_1, t_2, t_3 are positive integers, then it is known from lemma 3.4 that there are two pairs of integers (a_1, q_1), (a_2, q_2), satisfied

$$(a_1, q_1) = 1, (a_2, q_2) = 1, a_1 a_2 \neq 0,$$

$$1 \le q_i \ll \left(\frac{X^{1/3+\varepsilon}}{Y_i}\right)^2, |q_i\lambda_i\alpha - a_i| \ll X^{-1}\left(\frac{X^{1/3+\varepsilon}}{Y_i}\right)^2, \quad i = 1,2. \tag{6.4}$$

Note that $\alpha \ge u = \varsigma 2^{t_3} \ge \varsigma = X^{-7/9-\varepsilon}$,hence we have

$$\left|\frac{a_i}{\alpha}\right| \ll \left|\frac{a_i}{\lambda_i\alpha}\right| \ll q_i + \left(\frac{X^{1/3+\varepsilon}}{Y_i}\right)^2 \frac{1}{|\lambda_i\alpha|} \ll q_i, i = 2,3. \tag{6.5}$$

Further, divide the set $A(Y_1, Y_2, u)$ into smaller sets $A(Y_1, Y_2, u, P_1, P_2)$, where $P_i \le q_i < 2P_i$, $P_i \ll \left(\frac{X^{1/3+\varepsilon}}{Y_i}\right)^2$. Then there is

$$\begin{aligned}
&\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \\
&= \left|\frac{a_1(a_2 - q_2\lambda_2\alpha) + a_2(q_1\lambda_1\alpha - a_1)}{\lambda_2\alpha}\right| \\
&\ll P_1X^{-1}\left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2 + P_2X^{-1}\left(\frac{X^{1/3+\varepsilon}}{Y_1}\right)^2 \\
&\ll X^{-1}\left(\frac{X^{1/3+\varepsilon}}{Y_1}\right)^2\left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2 \ll X^{-1+2\sigma-\varepsilon} \\
&\ll X^{-8/9-\varepsilon} \ll N^{-(17/18)\cdot(8/9)-\varepsilon}.
\end{aligned} \tag{6.6}$$

Where $N^{17/18} \le X \le N$ .For equation (6.5) and $\alpha \in A(Y_1, Y_2, u, P_1, P_2)$, there is

$$|a_2q_1| \ll |q_1q_2\alpha| \ll uP_1P_2X^{\varepsilon}.$$

For sufficiently large $N^{(17/18)\cdot(8/9)} = q$, a/q converge on λ_1/λ_2 , by (6.6) there is

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| < \frac{1}{4q}.$$

Let $|a_2q_1|$ be taken with R different values, which are known by the pigeon nest principle and the best rational approximation principle

$$R \ll \frac{X^{\varepsilon}uP_1P_2}{q}.$$

From the classical upper bound estimation of the divisor function, it is known that each one $|a_3q_2|$ corresponds at most $\ll X^{\varepsilon}$, and $A(Y_1, Y_2, u, P_1, P_2)$ the length of the set does not exceed

$$\begin{aligned}
&\ll RX^{\varepsilon}\min\left(\frac{1}{P_1X}\left(\frac{X^{1/3+\varepsilon}}{Y_1}\right)^2, \frac{1}{P_2X}\left(\frac{X^{1/3+\varepsilon}}{Y_2}\right)^2\right) \\
&\ll \frac{X^{2\varepsilon}uP_1P_2}{q}\frac{X^{1/3+\varepsilon}}{Y_1}\frac{X^{1/3+\varepsilon}}{Y_2}\frac{1}{P_1^{1/2}P_2^{1/2}X} \\
&\ll \frac{uX^{1/3+5\varepsilon}}{qY_1^2Y_2^2}.
\end{aligned}$$

Calculation the integral on $A(Y_1, Y_2, u, P_1, P_2)$ below, we have

$$\begin{aligned}
&\int_{A(Y_1,Y_2,u,P_1,P_2)} |S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)|^2 K_\tau(\alpha)d\alpha \\
&\ll \min(\tau^2, y^{-2})Y_1^2Y_2^2X^{5/3}\int_{A(Y_1,Y_2,u,P_1,P_2)} d\alpha \\
&\ll \tau u^{-1}Y_1^2Y_2^2X^{5/3}\frac{X^{1/3+5\varepsilon}u}{qY_1^2Y_2^2} \\
&\ll \frac{\tau X^{2+5\varepsilon}}{q} \ll \tau X^{2-16\sigma+5\varepsilon}.
\end{aligned} \tag{6.7}$$

Next, the dichotomy is used to Y_1, Y_2, u, P_1, P_2 sum all possible values

$$\int_{\hat{c}} |S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{2-16\sigma+6\varepsilon}. \tag{6.8}$$

Certification. By Lemma 6.1 and 6.2. The following lemma is obtained.

Lemma 6.3 We have

$$\int_c |S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha) S_4(\lambda_5\alpha)|^2 K_\tau(\alpha)d\alpha \ll \tau X^{2-\sigma+\varepsilon}. \tag{6.9}$$

7. Proof of Theorem 1.2

Let $\tau = X^{-\delta}$ by the equations (2.4) and (2.5) have $v \in \mathbb{E}\left(\left[\frac{X}{2}, X\right]\right), I(v, X, \mathbb{R}) = 0$ and therefore have

$$\sum_{v\in\mathbb{E}([X/2,X])} I(v, X, \mathbb{R}) = \sum_{v\in\mathbb{E}([X/2,X])} I(v, X, \mathcal{M}) + I(v, X, c) + I(v, X, r) = 0. \tag{7.1}$$

It is known from Lemma 4.3 and 5.1

$$\left|\sum_{v\in\mathbb{E}([X/2,X])}\int_c S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha) S_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha\right| \gg \tau^2X^{1/2}E\left(\left[\frac{X}{2}, X\right]\right). \tag{7.2}$$

Derived from Cauchy's inequality and equation (2.1).

$$\left|\sum_{v\in\mathbb{E}([X/2,X])}\int_c S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)\right.$$
$$S_4(\lambda_5\alpha)K_\tau(\alpha)e(-v\alpha)d\alpha\Big|$$
$$\ll \tau^{1/2}\left(E\left(\left[\frac{X}{2},X\right]\right)\right)^{1/2}\left(\int_c\left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)\right.\right. \tag{7.3}$$
$$\left.\left.S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)\right|^2K_\tau(\alpha)d\alpha\right)^{1/2}$$

Combine equations (7.2) and (7.3) to obtain:

$$E\left(\left[\frac{X}{2},X\right]\right)\ll\tau^{-3}X^{-1}\int_c\left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)\right. \tag{7.4}$$
$$\left.S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)\right|^2K_\tau(\alpha)d\alpha.$$

Next, let's start the first part of proving theorem 1.2.Note that irrational numbers λ_1/λ_2 have rational approximation a/q fractions, $N=q^{(18/17)\cdot(9/8)}$ sum $N^{17/18}\le X\le N$, utilizing (7.4) and lemma 6.7 have

$$E\left(\left[\frac{X}{2},X\right]\right)\ll\tau^{-3}X^{-1}\tau X^{2-1/18+\varepsilon}\ll X^{1+2\delta-1/18+\varepsilon}. \tag{7.5}$$

Then, from (2.1) and (7.5), it is inferred

$$\begin{aligned}E(\mathcal{V},N,\delta)&\ll E\left(\left[N^{17/18},N\right]\right)+N^{17/18}\\&\ll\sum_{l=1}^{[(1/18)\log_2N]+1}E\left(\left[2^{-l}N,2^{1-l}N\right]\right)+N^{17/18}\\&\ll\sum_{l=1}^{[(1/18)\log_2N]+1}\left(2^{1-l}N\right)^{1+2\delta-1/18+2\varepsilon}+N^{17/18}\\&\ll N^{1+2\delta-1/18+3\varepsilon}.\end{aligned} \tag{7.6}$$

Obviously, irrational numbers λ_1/λ_2 rationally approximate the denominator of the fractional sequence q_j, and there is a sequence $N_j\to\infty$, so that the first part of theorem 1.2 is proved.

Let's prove the second part of theorem 1.2. Let it be any sufficiently large positive number, repeat the proof process in section 6, and then take it

$$\sigma=\chi \tag{7.7}$$

where is χ defined by equation (1.5). At this point, the expression for lemma 6.1 becomes

$$\int_{c'}\left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)\right. \tag{7.8}$$
$$\left.S_4(\lambda_5\alpha)\right|^2K_\tau(\alpha)d\alpha\ll\tau X^{2-\chi+\varepsilon}.$$

Repeating the proof process of lemma 6.2, equation (6.6) becomes

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2}-a_1q_2\right|\ll X^{-1+2\chi-\varepsilon}.$$

From equation (1.3), there are irrational numbers λ_1/λ_2 that satisfy a/q the rational approximation fraction

$$X^{(1-\omega)(1-2\chi)}\ll q\ll X^{1-2\chi}.$$

Then the formula (6.7) becomes

$$\int_{A(Y_1,Y_2,u,P_1,P_2)}\left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)\right|^2$$
$$K_\tau(\alpha)d\alpha$$
$$\ll\frac{\tau X^{2+5\varepsilon}}{q}\ll\tau X^{2+5\varepsilon}X^{-(1-\omega)(1-2\chi)}$$
$$\ll\tau X^{2-\chi+5\varepsilon}.$$

Equation (6.) in Lemma 6.28) becomes

$$\int_{\hat{c}}\left|S_3(\lambda_1\alpha)S_3(\lambda_2\alpha)S_3(\lambda_3\alpha)S_4(\lambda_4\alpha)S_4(\lambda_5\alpha)\right|^2 \tag{7.9}$$
$$K_\tau(\alpha)d\alpha\ll\tau X^{2-16\sigma+6\varepsilon}$$

Immediately after the transformation of equation (7.6), and the combination of equations (7.4), (7.8) and(7.9), the second part of theorem 1.2 is proved.

8. Conclusion

In this paper, the integral interval is divided into the major arc, the minor arc, and the trivial arc, which is proved by the Davenport-Heilbronn method *Cauchy' s* with the help of inequalities

$$I(v,X;\mathcal{M})\gg\tau^2X^{1/2},I(v,X;m)\gg\tau^2X^{1/2},$$
$$I(v,X;t)=o\left(\tau^2X^{1/2}\right).$$

That is, the integral on the main interval is the main term, and the integral on the coincidence interval and the trivial interval is the remainder, so as to obtain more accurate results.

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