

Mean Value Estimation of The Sum-of-divisors Function

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Abstract: The sum-of-divisor function is one of the important number theory functions, and the study of the properties of the sum-of-divisor function can provide more methods for solving some number theory problems. Since the value of the function is irregular with the change of the independent variables, the mean estimation of the sum-of-divisor function is usually used to deduce its own properties. Therefore, it is meaningful to study the mean estimation of the sum-of-divisor function and make gradual improvements to the rest of the terms. Through more in-depth study of power sum functions, the important properties of sum-of-divisor function can be obtained, which provides new ideas for solving more number theory problems. Let $\sigma_3(n)$ be the sum of all factors of n to the third power, $[x]$ be the integer part of x . Using the Dirichlet hyperbolic principle, the following results are obtained: the asymptotic formula for Sum-of-Divisors function $\sigma_3([x/n])$, and obtains the remainder of its asymptotic formula as $\Omega(x^2)$.

Keywords: Asymptotic formula; Sum-of-Divisors function; Dirichlet hyperbolic principle.

1. Introduction

Usually, $\varphi(n) = \sum_{1 \leq d \leq n, (d,n)=1} 1$ is called the Euler function.

$d(n) = \sum_{d|n} 1$ is the divisor function. In 1919, O. Bordellès, L.

Dai et al.^[1] used a new index pair method proposed by Bourgain^[2] to obtain: for $x \geq 3$, when $x \rightarrow \infty$, they proved

$$\begin{aligned} & \left(\frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + o(1) \right) x \log x \\ & \leq \sum_{n \leq x} \varphi \left(\left[\frac{x}{n} \right] \right) \\ & \leq \left(\frac{2629}{4009} \cdot \frac{1}{\zeta(2)} + \frac{1380}{4009} + o(1) \right) x \log x, \end{aligned}$$

and conjectured

$$\sum_{n \leq x} \varphi \left(\left[\frac{x}{n} \right] \right) = \left(\frac{1}{\zeta(2)} + o(1) \right) x \log x.$$

Since then, some scholars had improved the results^[3].

In 2020, Ma Jing and Sun Huayan^[8] by using the Van der Corput exponential pair theory^[9] proved that

$$\sum_{n \leq x} d \left(\left[\frac{x}{n} \right] \right) = x \sum_{m \geq 1} \frac{d(m)}{m(m+1)} + O_{\varepsilon} \left(x^{11/23+\varepsilon} \right)$$

for arbitrary $\varepsilon > 0$ and $x \rightarrow \infty$.

In 2020, Zhai Wenguang^[4] used Vinogradov's method of dealing with trigonometric sums and the theory of exponents^[9] to prove that

$$\sum_{n \leq x} \varphi \left(\left[\frac{x}{n} \right] \right) = \frac{x \log x}{\zeta(2)} + O \left(x (\log x)^{2/3} (\log \log x)^{1/3} \right)$$

for $x \rightarrow \infty$.

In 2021, Zhao Feng and Wu Jie^[10] demonstrated by refining the literature [4] that

$$\sum_{n \leq x} \sigma \left(\left[\frac{x}{n} \right] \right) = \frac{\pi^2}{6} x \log x + O \left(x (\log x)^{2/3} (\log_2 x)^{4/3} \right)$$

for $x \rightarrow \infty$.

In 2024, Yu Tingwei^[11] by using the Dirichlet hyperbolic principle proved that for $x \rightarrow \infty$

$$\sum_{n \leq x} \sigma_2 \left(\left[\frac{x}{n} \right] \right) = C_1(x) x^2 + O \left(x^{\frac{3}{2}} \right),$$

where $1 < C_1(x) < 2\zeta(2)$.

In this paper, using the Dirichlet hyperbolic principle, we prove the following theorem:

Theorem 1 (i) For $x \rightarrow \infty$, we have

$$S_{\sigma_3}(x) := \sum_{n \leq x} \sigma_3 \left(\left[\frac{x}{n} \right] \right) = A_1 x^3 + O(x^2), \quad (1.1)$$

Where $1 < A_1(x) < \frac{3}{2} \zeta(3)$, $A_1(x)$ denotes a constant that depends on x .

(ii) Let $M_{\sigma_3}(x)$ denote the remainder of $S_{\sigma_3}(x)$. For $x \rightarrow \infty$, we have

$$M_{\sigma_3}(x) = \Omega(x^2). \quad (1.2)$$

2. Preliminary Lemmas

Lemma 2.1 Let p be prime, and $n = p_1^{l_1} p_2^{l_2} p_3^{l_3} \cdots p_k^{l_k} = \prod p^l$, we have

$$\frac{\sigma_3(n)}{n^3} = \prod_{p|n} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3l}} \right). \quad (2.1)$$

Proof. From the definition of the multiplicative function, we know that $\sigma_3 = \sum_{d|n} d^3$ is the multiplicative function.

Since $n = p_1^{l_1} p_2^{l_2} p_3^{l_3} \cdots p_k^{l_k} = \prod p^l$, we have

$$\begin{aligned} \sigma_3(n) &= \sigma_3(p_1^{l_1} p_2^{l_2} \cdots p_k^{l_k}) \\ &= \sigma_3(p_1^{l_1}) \sigma_3(p_2^{l_2}) \cdots \sigma_3(p_k^{l_k}). \end{aligned}$$

So

$$\begin{aligned} \frac{\sigma_3(n)}{n^3} &= \frac{\sigma_3(p_1^{l_1}) \sigma_3(p_2^{l_2}) \cdots \sigma_3(p_k^{l_k})}{p_1^{3l_1} p_2^{3l_2} \cdots p_k^{3l_k}} \\ &= \prod_{p|n} \frac{1 + p^3 + \cdots + p^{3l}}{p^l} \\ &= \prod_{p|n} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3l}} \right). \end{aligned}$$

This completes the proof of lemma 2.1.

Lemma 2.2 (i) Let $\psi(x) := x - [x] - (1/2)$. When $x \geq 2$ and $1 \leq a \leq x^{(1/5)}$, we have

$$\begin{aligned} \sum_{n \leq x} \sigma_3(n) &= \frac{1}{4} \zeta(4) x^4 - \frac{a^3 + [a]^4 - 2[a]^3 - [a]}{12} x \\ &\quad - \Delta(x, a) + O\left(\frac{x}{a}\right), \end{aligned} \quad (2.2)$$

Where

$$\begin{aligned} \Delta_1(x, a) &:= \frac{3}{2} \sum_{d \leq (x/a)} \frac{x^2}{d^2} \psi^2\left(\frac{x}{d}\right) - \frac{1}{8} \sum_{d \leq (x/a)} \frac{x^2}{d^2} - \sum_{d \leq (x/a)} \frac{x}{d} \psi^3\left(\frac{x}{d}\right) \\ &\quad + \frac{1}{4} \sum_{d \leq (x/a)} \frac{x}{d} \psi\left(\frac{x}{d}\right) - \frac{3}{4} \sum_{d \leq (x/a)} \frac{x}{d} \\ &\quad + \sum_{d \leq (x/a)} \frac{x^3}{d^3} \psi\left(\frac{x}{d}\right). \end{aligned} \quad (2.3)$$

(ii) For $x \rightarrow \infty$, we have

$$\sum_{n \leq x} \sigma_3(n) = \frac{1}{4} \zeta(4) x^4 + O(x^3). \quad (2.4)$$

Proof. According to the Dirichlet hyperbolic principle and $\sigma_3(n) = \sum_{d|n} m^3$, we can have

$$\sum_{n \leq x} \sigma_3(n) = \sum_{d|n} m^3 = S_1 + S_2 - S_3. \quad (2.5)$$

Firstly, since $|\psi(t)| \leq 1$ for all t , we have

$$\begin{aligned} S_1 &= \sum_{d \leq x/a} (1^3 + 2^3 + \cdots + [x/d]^3) \\ &= \frac{1}{4} \sum_{d \leq x/a} ([x/d]^4 + 3[x/d]^3 + [x/d]^2) \\ &= \frac{1}{4} \sum_{d \leq x/a} \left(\frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{1}{2} \right)^4 + \frac{1}{2} \sum_{d \leq x/a} \left(\frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{1}{2} \right)^3 \\ &\quad + \frac{1}{4} \sum_{d \leq x/a} \left(\frac{x}{d} - \psi\left(\frac{x}{d}\right) - \frac{1}{2} \right)^2 \\ &= \frac{1}{4} \sum_{d \leq x/a} \frac{x^4}{d^4} + \frac{3}{2} \sum_{d \leq x/a} \frac{x^2}{d^2} \psi^2\left(\frac{x}{d}\right) - \frac{1}{8} \sum_{d \leq x/a} \frac{x^2}{d^2} \\ &\quad - \sum_{d \leq x/a} \frac{x}{d} \psi^3\left(\frac{x}{d}\right) + \frac{1}{4} \sum_{d \leq x/a} \frac{x}{d} \psi\left(\frac{x}{d}\right) - \frac{3}{4} \sum_{d \leq x/a} \frac{x}{d} \\ &\quad - 3 \sum_{d \leq x/a} \frac{x}{d} \psi^2\left(\frac{x}{d}\right) + \frac{1}{4} \sum_{d \leq x/a} \psi^4\left(\frac{x}{d}\right) - \frac{1}{8} \sum_{d \leq x/a} \psi^2\left(\frac{x}{d}\right) \\ &\quad + \frac{3}{8} \sum_{d \leq x/a} 1 + \sum_{d \leq x/a} \psi^3\left(\frac{x}{d}\right) + \frac{1}{4} \sum_{d \leq x/a} \frac{x^3}{d^3} \psi\left(\frac{x}{d}\right) + \frac{3}{8} \\ &= \frac{1}{4} \zeta(4) x^4 - \frac{1}{12} x a^3 - \Delta_1(x, a) + O\left(\frac{x}{a}\right). \end{aligned} \quad (2.6)$$

where $\Delta_1(x, a)$ is defined by (2.3).

Secondly, we have

$$S_2 = \sum_{m \leq a} m^3 \left[\frac{x}{m} \right] = \frac{2[a]^3 + 3[a]^2 + [a]}{6} x + O(a^4), \quad (2.7)$$

$$\begin{aligned} S_3 &= \sum_{d \leq (x/a)} (1^3 + 2^3 + \cdots + [a]^3) \\ &= (1^3 + 2^3 + \cdots + [a]^3) \left[\frac{x}{a} \right] \\ &= \frac{[a]^4 + 2[a]^3 + [a]}{4} x + O(a^4). \end{aligned} \quad (2.8)$$

Inserting (2.6), (2.7), (2.8) into (2.5) and using $a^4 \leq (x/a)$, we can obtain (2.2).

Taking $a = 1$, we have

$$\Delta_1(x, 1) \ll x^3 \sum_{d \leq x} \frac{1}{d^3} + x^2 \sum_{d \leq x} \frac{1}{d^2} + x \log x \ll x^3.$$

So, we can get (2.4). This completes the proof of lemma 2.2.

3. Proof of the Theorem 1

3.1. Proof of the Theorem 1(i)

Proof. Let $d = \left[\frac{x}{n} \right]$, we have $(x/n) - 1 < d \leq (x/n)$, $x/(d+1) < n \leq (x/d)$ and $\sigma_3(0) = 0$. We have, with the Dirichlet hyperbolic principle

$$\begin{aligned} S_{\sigma_3}(x) &= \sum_{d \leq x} \sigma_3(d) \sum_{(x/(d+1)) < n \leq (x/d)} 1 \\ &= \sum_{d \leq x} \sigma_3(d) - \sum_{d \leq x, d \geq 2} \sigma_3(d-1) \\ &= \sum_{d \leq x} (\sigma_3(d) - \sigma_3(d-1)) \end{aligned}$$

$$= S_1(x, \sigma_3) + S_2(x, \sigma_3) - S_3(x, \sigma_3), \quad (3.1)$$

where

$$\begin{aligned} S_1(x, \sigma_3) &:= \sum_{d \leq \sqrt{x}, dn \leq x} (\sigma_3(d) - \sigma_3(d-1)), \\ S_2(x, \sigma_3) &:= \sum_{n \leq \sqrt{x}, dn \leq x} (\sigma_3(d) - \sigma_3(d-1)), \\ S_3(x, \sigma_3) &:= \sum_{d \leq \sqrt{x}, n \leq \sqrt{x}} (\sigma_3(d) - \sigma_3(d-1)). \end{aligned}$$

First, we estimate $S_1(x, \sigma_3)$. For $S_1(x, \sigma_3)$, we have

$$\begin{aligned} S_1(x, \sigma_3) &= \sum_{d \leq \sqrt{x}} (\sigma_3(d) - \sigma_3(d-1)) \left\lfloor \frac{x}{d} \right\rfloor \\ &= x \sum_{d \leq \sqrt{x}} \frac{\sigma_3(d) - \sigma_3(d-1)}{d} \\ &\quad + O\left(\sum_{d \leq \sqrt{x}} |\sigma_3(d) - \sigma_3(d-1)|\right). \end{aligned} \quad (3.2)$$

With the help of Lemma 2(ii), we can obtain

$$\begin{aligned} &x \sum_{d \leq \sqrt{x}} \frac{\sigma_3(d) - \sigma_3(d-1)}{d} \\ &= x \sum_{d \leq \sqrt{x}} \frac{\sigma_3(d)}{d(d+1)} = x \sum_{d \leq \sqrt{x}} \frac{\sigma_3(d)}{d^2} - x \sum_{d \leq \sqrt{x}} \frac{\sigma_3(d)}{d^2(d+1)} \\ &= x \int_{1-}^{\sqrt{x}} t^{-2} d\left(\frac{1}{4} \zeta(4) t^4 + O(t^3)\right) + O(x) \\ &= \frac{1}{2} \zeta(4) x^2 + O\left(x^{\frac{3}{2}}\right). \end{aligned} \quad (3.3)$$

$$\sum_{d \leq \sqrt{x}} |\sigma_3(d) - \sigma_3(d-1)| \ll \sum_{d \leq \sqrt{x}} \sigma_3(d) \ll x^2. \quad (3.4)$$

Inserting (3.3), (3.4) into (3.2), we obtain

$$S_1(x, \sigma_3) \ll x^2. \quad (3.5)$$

Secondly, we estimate $S_2(x, \sigma_3)$. Let $[x/n] = \prod p^\alpha$.

With the help of Lemma 1, we have

$$\begin{aligned} S_2(x, \sigma_3) &= \sum_{n \leq \sqrt{x}} \sigma_3([x/n]) \\ &= \sum_{n \leq \sqrt{x}} [x/n]^3 \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \\ &= x^3 \sum_{n \leq \sqrt{x}} \frac{1}{n^3} \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \\ &\quad - 3x^2 \sum_{n \leq \sqrt{x}} \frac{1}{n^2} \left\{ \frac{x}{n} \right\} \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \\ &\quad + 3x \sum_{n \leq \sqrt{x}} \frac{1}{n} \left\{ \frac{x}{n} \right\}^2 \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \\ &\quad - \sum_{n \leq \sqrt{x}} \left\{ \frac{x}{n} \right\}^3 \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right). \end{aligned} \quad (3.6)$$

For the first item of (3.6), we have

$$\begin{aligned} &x^3 \sum_{n \leq \sqrt{x}} \frac{1}{n^3} \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \\ &= \left(A'(x) + \frac{1}{2^3} A''(x) + \frac{1}{3^3} A'''(x) + \cdots + \frac{1}{(\sqrt{x})^3} A^*(x) \right) x^3 \\ &= A_1(x) x^3, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} A'(x) &= \prod_{p \mid [x]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right), \\ A''(x) &= \prod_{p \mid [x/2]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right), \\ &\quad \dots \\ A^*(x) &= \prod_{p \mid [x/\sqrt{x}]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right), \end{aligned}$$

$A'(x), A''(x), \dots, A^*(x)$ are constant that depend on x .

With the help of literature [12], we have

$$1 \leq \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) < \zeta(3),$$

so

$$1 \leq A'(x), A''(x), \dots, A^*(x) < \zeta(3).$$

Since

$$\begin{aligned} A_1(x) &< \zeta(3) \left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{(\sqrt{x})^3}\right) \\ &< \zeta(3) \left(1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}+1}\right) \\ &< \frac{3}{2} \zeta(3), \end{aligned}$$

we have $1 < A_1(x) < \frac{3}{2} \zeta(3)$.

For the second item of (3.6), we have

$$-3x^2 \sum_{n \leq \sqrt{x}} \frac{1}{n^2} \left\{ \frac{x}{n} \right\} \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \ll x^{\frac{3}{2}}. \quad (3.8)$$

For the third item of (3.6), we have

$$\begin{aligned} &3x \sum_{n \leq \sqrt{x}} \frac{1}{n} \left\{ \frac{x}{n} \right\}^2 \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \\ &\ll x \sum_{n \leq \sqrt{x}} \frac{1}{n} \ll x \log x. \end{aligned} \quad (3.9)$$

For the fourth item of (3.6), we have

$$- \sum_{n \leq \sqrt{x}} \left\{ \frac{x}{n} \right\}^3 \prod_{p \mid [x/n]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3\alpha}}\right) \ll x. \quad (3.10)$$

Inserting (3.7), (3.8), (3.9), (3.10) into (3.6), we can obtain

$$S_2(x, \sigma_3) = A_1(x)x^3 + O\left(x^{\frac{3}{2}}\right), \quad (3.11)$$

where $1 < A_1(x) < \frac{3}{2}\zeta(3)$.

Finally, we estimate $S_3(x, \sigma_3)$. Let $[x/n] = \prod p^\alpha$. With the help of Lemma 1, we obtain

$$\sigma_3([x]) = [x]^3 \prod_{p|[x]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3l}}\right).$$

$S_3(x, \sigma_3)$ can write

$$\begin{aligned} S_3(x, \sigma_3) &= [\sqrt{x}] \sigma_3([\sqrt{x}]) \\ &= \prod_{p|[x]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3l}}\right) [\sqrt{x}]^4 \\ &= \prod_{p|[x]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3l}}\right) x^2 + O(x). \end{aligned} \quad (3.12)$$

Let

$$\prod_{p|[x]} \left(1 + \frac{1}{p^3} + \frac{1}{p^6} + \cdots + \frac{1}{p^{3l}}\right) = A_2(x),$$

where $A_2(x)$ is a constant that depends on x . Therefore, we have

$$S_3(x, \sigma_3) = A_2(x)x^2 + O(x). \quad (3.13)$$

Inserting (3.5), (3.11), (3.13) into (3.1), we have Theorem 1 (i).

3.2. Proof of the Theorem 1(ii)

Proof. Let $M_{\sigma_3}(x)$ is the remainder of (1.1) and denote $M_{\sigma_3}^*(x) := \max\{|M_{\sigma_3}(p)|, |M_{\sigma_3}(p-1)|\}$. For any odd prime p , we have

$$\begin{aligned} &\sum_{d|p} (\sigma_3(d) - \sigma_3(d-1)) \\ &= S_{\sigma_3}(p) - S_{\sigma_3}(p-1) \\ &= A_1 p^3 + M_{\sigma_3}(p) - A_1 (p-1)^3 - M_{\sigma_3}(p-1) \\ &\geq M_{\sigma_3}(p) - M_{\sigma_3}(p-1) \geq -2M_{\sigma_3}^*(p), \end{aligned} \quad (3.14)$$

where $M_{\sigma_3}^*(x) := \max\{|M_{\sigma_3}(p)|, |M_{\sigma_3}(p-1)|\}$.

On the other hand, we have

$$\begin{aligned} &\sum_{d|p} (\sigma_3(d) - \sigma_3(d-1)) \\ &= \sigma_3(p) - \sigma_3(p-1) + 1 \\ &\leq p^3 + 1 - \left((p-1)^3 + \frac{1}{8}(p-1)^3 + 2^3 + 1\right) + 1 \end{aligned}$$

$$< \frac{27}{8} p^2, \quad (3.15)$$

Combining (3.14) with (3.15), we get

$$M_{\sigma_3}^*(p) > -\frac{27}{16} p^2.$$

This completes the proof of Theorem 1(ii).

4. Conclusion

Number theory function is an important concept in number theory, which is the basis for studying other number theory problems. The sum-of-divisors function is one of the important number theory functions, and we usually use the mean estimation of the sum-of-divisors function to deduce its own properties, and the study of the properties of the sum-of-divisors function can provide more methods for solving some number theory problems. In this paper, we mainly use the Dirichlet hyperbolic principle to obtain the following results: the asymptotic formula for sum-of-divisors function $\sigma_3([x/n])$, and obtains the remainder of its asymptotic formula as $\Omega(x^2)$. Through more in-depth study of power sum functions, we obtain the important properties of power sum functions, which provide new ideas for solving more number theory problems.

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