

Financial Asset Price Volatility Models and Empirical Analysis Based on Stochastic Differential Equations

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Abstract. To accurately characterize the price volatility characteristics of financial assets, this paper uses the closing prices of the CSI 300 Index for 1488 trading days from 2018 to 2023 as a sample. Two types of stochastic differential equation models, basic and extended, are constructed. The basic model is centered on geometric Brownian motion and consists solely of the drift coefficient μ and the volatility coefficient σ . The extended model introduces dynamic jumps and stochastic volatility, accounting for both continuous fluctuations and extreme jumps. Using maximum likelihood estimation (MLE) for parameter estimation, data simulations show that the base model has $\mu=0.062$ (annualized return 6.2%, $p<0.01$) and $\sigma=0.185$ (annualized $\pm$ -volatility 18.5%, $p<0.001$). All parameters of the extended model are significant ($p<0.05$), with an annualized jump of 2.4 times, a 5% increase in the probability of a jump for every 1-point increase in the VIX, and a 1.7-month return to normal volatility. Fitting and forecasting simulations validate the advantages of the extended model: the fitted RMSE is reduced by 40% compared to the base model (0.035 to 0.021), and the out-of-sample forecast RMSE is reduced by 40.48% (0.042 to 0.025). The fitted error for the extreme volatility point in October 2022 is less than 2%. Research shows that an expanded model incorporating jumps and stochastic volatility better aligns with the fluctuation patterns of the A-share market and can provide support for risk management.

Keywords: Stochastic Differential Equations; Financial Asset Price Fluctuations; Geometric Brownian Motion.

1. Introduction

Currently, global financial markets operate in a highly complex and uncertain environment, and the volatility of financial asset prices exhibits significant nonlinearity and time-varying characteristics. From the global stock market circuit breakers triggered by the COVID-19 pandemic in 2020, to the drastic bond market correction caused by the Federal Reserve's aggressive interest rate hikes in 2022, to the price volatility of risky assets triggered by the banking crisis in 2023, a series of frequent black swan and gray rhino events have made it even more difficult to capture the volatility patterns of financial asset prices. Dynamic adjustments in macroeconomic fundamentals (such as cyclical fluctuations in GDP growth and structural shifts in inflation), periodic shifts in macroeconomic policies (including interest rate adjustments in monetary policy and marginal tightening or easing of industry regulations), and irrational fluctuations in market sentiment (such as sudden rises and falls in investor confidence and panic trading in extreme market conditions) collectively constitute multi-dimensional drivers of asset prices, leading to a continuous increase in the frequency and magnitude of price fluctuations in core financial assets such as stocks, bonds, and foreign exchange [1]. These drastic and unpredictable price fluctuations not only complicate investor asset allocation decisions—for example, traditional mean-variance models based on historical data struggle to cope with extreme volatility—but also place higher demands on financial institutions' risk hedging mechanisms and liquidity management capabilities [2]. Some institutions have fallen into liquidity crises due to their failure to accurately anticipate volatility risks. Against this backdrop, developing quantitative models that accurately capture the patterns of financial asset price fluctuations and adapt to complex market environments has become a critical issue urgently needed in the field of financial engineering [3]. In existing research, traditional ARCH models, while able to capture volatility clustering, struggle to characterize the continuous dynamics of price fluctuations. Simple geometric Brownian motion models, while based on stochastic process theory, ignore the ubiquitous jumps and time-varying

volatility in the market, limiting model fitting and forecasting accuracy [4]. Addressing this research gap, this paper, using stochastic differential equations as a core modeling tool, investigates the quantitative characterization of financial asset price fluctuations, aiming to improve the accuracy of volatility characterization through model optimization. The research mainly includes three parts: First, a basic stochastic differential equation model of financial asset price fluctuations is constructed based on geometric Brownian motion. The economic connotations of the drift term (reflecting the long-term expected return of the asset) and the diffusion term (reflecting the intensity of the asset's random fluctuations) in the model are clarified, laying the foundation for subsequent expansion [5]. Second, combining the common jump fluctuations in financial markets (such as price jumps caused by sudden policy releases) with the characteristics of volatility clustering (such as persistently high volatility in a bear market), a Poisson jump process is introduced to characterize discontinuous fluctuations, and a stochastic volatility factor is introduced based on the Heston model to construct an extended volatility model that is more in line with the actual market. Third, the A-share market CSI 300 Index (daily data from 2018-2023) is selected as the research sample of representative financial assets. Empirical tests are conducted using high-frequency trading data. The model parameters are estimated using the maximum likelihood estimation method [6]. The fitting effect of the basic model and the extended model is compared using indicators such as root mean square error (RMSE), mean absolute error (MAE), and log-likelihood. The practical application value of the model is verified through out-of-sample predictions. Methodologically, the paper first systematically review the application progress and research gaps of stochastic differential equations in financial volatility modeling through a literature review to clarify the research positioning of this paper. Secondly, the paper employ mathematical derivation to construct and define parameters for two types of models, ensuring the theoretical rigor of the models [7]. Finally, using Python data analysis tools, the paper conduct quantitative model testing and analysis through data cleaning (missing value interpolation and outlier removal), parameter estimation, and indicator calculation, providing reliable data support for the research conclusions.

2. Construction of a Financial Asset Price Volatility Model Based on Stochastic Differential Equations

2.1. Basic Model Construction

Based on the theoretical core of geometric Brownian motion, the paper construct a basic stochastic differential equation model for the price fluctuations of a single financial asset. This model is a classic framework for characterizing continuous fluctuations, and its parameters have clear economic significance, facilitating subsequent empirical testing [8]. We select asset price S_t as the dependent variable (yuan) and time t as the independent variable (years). We introduce a drift term to reflect the long-term return trend of the asset, and a diffusion term to reflect the characteristics of random fluctuations. From the perspective of financial asset pricing logic, asset returns are generally positively correlated with prices (the higher the price, the greater the absolute value of the return per unit of capital), and the intensity of fluctuations changes proportionally with price growth (the absolute amount of a single fluctuation of a high-priced asset is higher). Based on this, the stochastic differential equation form of the basic model is as follows:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (1)$$

In the above formula, the value range and economic meaning of each parameter need to be calibrated in combination with the actual market: μ is the drift coefficient, which represents the expected return of the asset per unit time (year⁻¹). Its value is determined by the macroeconomic environment (such as GDP growth rate), industry prosperity (such as the PMI index), and the asset's own profitability (such as ROE). Taking the CSI 300 Index as an example, the annualized return from 2018 to 2023 was approximately 6.2%. Therefore, the reasonable value range of μ is usually between 2% and 8%. If $\mu > 0$, it indicates that the asset has a long-term upward trend, which is consistent

with the return characteristics of most equity assets. σ is the diffusion coefficient, that is, the asset price volatility (year^{-1/2}), which reflects the average intensity of price fluctuations per unit time. The annualized volatility of the CSI 300 Index from 2018 to 2023 was approximately 18%. Therefore, the common value of σ is between 12% and 25%. The larger σ is, the more severe the short-term price fluctuations. For example, the volatility of the ChiNext Index is usually higher than that of the CSI 300 Index, and σ can reach more than 30%. dW_t is the increment of the standard Wiener process, satisfying $dW_t \sim N(0, dt)$, and the increments at different times are independent of each other. For example, dW_{t_1} at time t_1 is equal to the independent values of dW_{t_2} at time t_2 ($t_1 \neq t_2$) accurately capture the randomness of price fluctuations, avoiding the appearance of predictable patterns.

Integrating both sides of the above equation and combining it with the properties of the Wiener process ($\int_0^t W_s dW_s = \frac{1}{2}(W_t^2 - t)$), the paper can derive the explicit solution for the asset price using Ito's lemma:

$$S_t = S_0 \exp \left[\left(\mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t \right] \quad (2)$$

In the above formula, S_0 is the initial asset price (e.g., the price at the beginning of the year), and the $-\frac{1}{2}\sigma^2$ within the exponential term is a volatility correction term. This correction stems from the difference between stochastic integration and ordinary calculus [9]. If this correction term is ignored, the expected return obtained by directly taking the logarithm of the basic model equation will systematically underestimate by $\frac{1}{2}\sigma^2$. For example, for $\sigma=20\%$, the underestimation is approximately 2% ($\frac{1}{2} \times 0.2^2 = 0.02$), significantly affecting the accuracy of the return calculation. Therefore, this correction term ensures that the expected return of S_t , $E[S_t] = S_0 e^{\mu t}$, is consistent with μ , avoiding the systematic underestimation of returns due to the diffusion term.

2.2. Extended Model Construction

To address the limitations of the basic model's "constant volatility and no jumps," and by integrating the "time-varying volatility and discontinuous jumps" characteristics of actual market fluctuations, the paper extend the model along two key dimensions to ensure it can capture both typical fluctuations and extreme scenarios.

2.2.1 Introduction and Parameter Design of Jump Processes

Considering price jumps triggered by major events (such as policy changes, disappointing earnings, and geopolitical conflicts), these jumps typically cause prices to deviate from their continuous path for a short period of time, which the basic model cannot capture [10]. To accurately capture jump characteristics, the paper first need to address two core questions: "When does the jump occur?" and "How large is the jump?" Regarding the timing of the jump, the paper assume that the jump intensity varies dynamically with market uncertainty. We introduce a state variable Z_t to reflect the level of market risk. The jump intensity λ_t is positively correlated with Z_t , as shown in the following relationship:

$$\lambda_t = \lambda_0 + \kappa Z_t \quad (3)$$

In the above formula, parameter calibration requires integration with historical jump data: λ_0 is the base jump intensity (year⁻¹), representing the probability of a jump occurring in the absence of major events. By counting the number of times the CSI 300 Index experienced a single-day increase or decrease of more than 5% from 2018 to 2023 (a total of 12 times), the paper calculated that $\lambda_0 \approx 2.4$ (i.e., an average of approximately 2.4 jumps per year); $\kappa > 0$ is the sensitivity coefficient (year⁻¹ · point⁻¹). Regression analysis yielded $\kappa \approx 0.05$, indicating that for every 1-point increase in the VIX index, the jump intensity increases by 0.05, consistent with the characteristic that "higher market risk leads to more frequent jumps."

2.2.2 Introduction of Stochastic Volatility and Equation Construction

Considering the time-varying nature of volatility—in real markets, volatility is not constant but rather exhibits a "clustering" pattern (periods of high volatility are often followed by periods of high volatility, and periods of low volatility are often followed by periods of low volatility). For example, A-share market volatility in 2022 remained consistently higher than in 2021, and volatility in Q1 of 2023 was higher than in Q2. Drawing on the Heston model but simplifying the parameter structure (reducing the number of parameters to reduce estimation difficulty), the paper assume that volatility σ_t^2 (i.e., variance) follows a mean-reverting process, meaning that volatility converges to its mean $\bar{\sigma}^2$ in the long term and deviates from the mean in the short term due to random shocks. The stochastic differential equation is as follows:

$$d\sigma_t^2 = \theta(\bar{\sigma}^2 - \sigma_t^2)dt + \eta\sigma_t dV_t \quad (4)$$

In the above formula, the economic significance and calibration basis of the parameters are as follows: $\theta > 0$ is the mean reversion rate (years⁻¹), reflecting the speed of volatility's return to the mean. Fitting the volatility data of the CSI 300 Index from 2018 to 2023 yields $\theta \approx 5$, indicating the time it takes for volatility to return to halfway after a deviation from the mean, consistent with the market characteristic that volatility typically returns to normal within 2-3 months. $\bar{\sigma}^2$ is the long-term mean volatility (years⁻¹). Calculated values of $\bar{\sigma}^2 \approx 0.0324$ (i.e., $\bar{\sigma} \approx 18\%$) are consistent with the long-term mean annualized volatility of the CSI 300 Index. η is the volatility of volatility (years^{-1/2}). Fitting yields $\eta \approx 0.3$, reflecting the intensity of volatility fluctuations. A larger η indicates more drastic short-term volatility fluctuations.

2.2.3 Integration and Discretization of the Extended Model

By integrating the jump process and stochastic volatility into the basic model, the resulting stochastic differential equation for the extended model is as follows. This model simultaneously incorporates the three core features of "continuous volatility + stochastic volatility + dynamic jumps," fully aligning with actual market fluctuation patterns:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t + S_t J_t dN_t \quad (5)$$

In the above formula, the newly added variables must be explicitly defined to ensure the model's operability: N_t is a Poisson process with strength λ_t , satisfying $P(N_t = k) = \frac{(\lambda_t t)^k e^{-\lambda_t t}}{k!}$ ($k = 0, 1, 2, \dots$), $dN_t = 1$ indicates a jump at time t , and $dN_t = 0$ indicates no jump; S_t is the asset price immediately before the jump (i.e., $S_t = \lim_{s \rightarrow t^-} S_s$). This ensures that the jump amplitude is calculated based on the price before the jump, not the price after the jump, consistent with the real market characteristic of "a jump occurs instantaneously, with the price jumping from S_t to $S_t = S_t(1 + J_t)$."

Since stochastic differential equations cannot be directly applied to empirical data (which are discrete daily data), the extended model must be discretized to facilitate subsequent parameter estimation and fit testing. Selecting a time step Δt (for daily data, $\Delta t = 1/252$, assuming 252 trading days per year), and combining the Euler discretization method, the extended model equation is converted into a discrete form as follows:

$$\Delta S_t = \mu S_{t-1} \Delta t + \sigma_{t-1} S_{t-1} \Delta W_t + S_{t-1} J_t \Delta N_t \quad (6)$$

In the above formula, the discrete variables and distribution characteristics are clear: $\Delta S_t = S_t - S_{t-1}$ is the price difference (in RMB) between day t and day $t - 1$, directly corresponding to the daily price fluctuations in the empirical data; $\Delta W_t \sim N(0, \Delta t)$ simulates continuous price fluctuations by generating random numbers that conform to this distribution; ΔN_t follows a Bernoulli distribution with parameter $\lambda_{t-1} \Delta t$ (i.e., $P(\Delta N_t = 1) = \lambda_{t-1} \Delta t$, $P(\Delta N_t = 0) = 1 - \lambda_{t-1} \Delta t$), simulating whether a jump occurs.

3. Empirical Analysis Design

3.1. Data Selection and Preprocessing

The study uses the CSI 300 Index as a sample (covering more than 60% of the A-share market value, a decline of 21.63% in 2022, with a deviation of less than 1% from the average of the entire market, and is more representative than the ChiNext Index), and selects the closing prices of 1,488 trading days from 2018 to 2023 (from Wind and the official website of the CSI Index). This period covers multiple scenarios such as a bear market (a decline of 25.31% in 2018), a volatile bull market (an increase of more than 73% from 2019 to 2020), and a stable period (an average daily fluctuation of 1.05% in 2021). The preprocessing process consists of two steps: first, linear interpolation is used to fill in missing values for 12 holidays (for example, the 2020 Spring Festival had an error rate of less than 2%, which is better than the mean-filling method); second, outliers are detected using the 3σ criterion, and the true fluctuation values caused by five events (such as the global circuit breakers in March 2020) are confirmed and retained [11]. One transmission error value is corrected, resulting in 1488 valid observations.

3.2. Parameter Estimation Method Selection

Comparison between the Maximum Likelihood Estimation (MLE) and the Generalized Method of Moments (GMM): MLE constructs a mixture distribution likelihood function, derives the log-likelihood function, and uses the L-BFGS-B algorithm to find the optimal parameters. This method is efficient but relies on distributional assumptions. GMM uses moment conditions for estimation, requiring no distributional assumptions but with limited accuracy. Considering the data characteristics (daily high frequency and slight non-normality), the MLE was selected: First, the introduction of jump terms makes the data closer to a mixed distribution; second, simulations show that the MLE volatility estimation error is 3.2%, lower than the GMM's 5.8%, meeting the accuracy requirements.

3.3. Determination of Model Verification Metrics

RMSE, which amplifies extreme errors, reflects the model's predictive ability for extreme fluctuations. Smaller values are better, making it suitable for risk management. MAE, which calculates the mean absolute error, reflects the average fit under normal fluctuations. Smaller values.

indicate greater stability, making it suitable for daily forecasting. LL, which sums the logarithmic probability density, reflects statistical explanatory power and is unaffected by price magnitude [12]. Larger values indicate higher goodness of fit, providing a theoretical basis for model selection.

4. Empirical Results and Analysis

4.1. Descriptive Statistical Analysis of Data

Table 1 shows that the average daily closing price of the CSI 300 Index from 2018 to 2023 was 3856.72 points, with a standard deviation of 528.36 points, indicating significant volatility. The skewness is -0.23 (slightly left-skewed, with a higher concentration of below-mean trading days), and the kurtosis is 4.15 (peaked, with a higher frequency of extreme fluctuations than in a normal distribution). Regarding extreme values, the highest price was 5441.80 points (February 18, 2021, during pre-holiday sentiment), the lowest price was 2271.64 points (October 24, 2022, during the interest rate hike and the impact of the epidemic), and the range was 3170.16 points. From 2018 to 2022, the market cycle showed a "decline-rise-oscillation-decline" pattern, with stability in 2023, providing evidence that the model accounts for both "continuous fluctuations and jumps."

Table 1. Descriptive Statistics of Daily Closing Prices of the CSI 300 Index from 2018 to 2023

statistical Indicators	Numerical	Unit	Analysis of differences from normal distribution
Sample Size	1488	Piece	-
Mean	3856.72	Point	-
Standard Deviation	528.36	Point	-
Skewness	-0.23	Dimensionless Quantity	Left-skewed (normal distribution = 0)
Kurtosis	4.15	Dimensionless Quantity	peaked (normal distribution = 3)
Minimum	2271.64	Point	-
Maximum	5441.8	Point	-
Range	3170.16	Point	-
Median	3812.45	Point	Close to the mean, slightly skewed to the left
25th Percentile	3425.18	Point	-
75th Percentile	4289.36	Point	-

4.2. Parameter Estimation Results

Table 2 shows the parameter estimation results for the base model and the extended model using maximum likelihood estimation. The base model only includes the drift coefficient μ and the volatility coefficient σ . The estimated value of μ is 0.062 ($t=3.85$, $p<0.01$), indicating a statistically significant annualized expected return of 6.2% for the CSI 300 Index. The estimated value of σ is 0.185 ($t=12.63$, $p<0.001$), corresponding to an annualized volatility of 18.5%, which is consistent with historical market volatility. The extended model adds parameters related to jumps and stochastic volatility, all of which pass significance tests ($p < 0.05$): the base jump intensity λ_0 is 2.4 (2.4 annualized jumps), and the VIX sensitivity κ is 0.05, indicating that the probability of a jump increases by 5% for every 1-point increase in market risk; the average jump amplitude α is -0.03 (a single jump results in a 3% drop), confirming that negative events dominate jumps; the mean volatility reversion rate θ is 5.0, indicating that volatility returns to normal 1.7 months after deviations from the mean; the long-term mean volatility $\bar{\sigma}$ is 0.18, close to σ in the base model, but the volatility of the added volatility η is 0.3, reflecting the time-varying nature of volatility. The extended model parameters are all significant and consistent with economic logic, demonstrating the rationale for introducing jumps and stochastic volatility.

Table 2. Parameter Estimation Results for the Base and Extended Models

model Type	parameter	Estimated value	Standard error	t statistic	p-value	Annualized conversion value
Basic Model	μ	0.062	0.016	3.85	0.0002	6.20%
	σ	0.185	0.014	12.63	<0.0001	18.50%
Extended Model	μ	0.058	0.015	3.87	0.0002	5.80%
	σ	0.172	0.013	13.23	<0.0001	17.20%
	λ	2.4	0.52	4.62	<0.0001	2.4 times/year
	κ	0.05	0.012	4.17	<0.0001	0.05/point
	α	-0.03	0.008	-3.75	0.0002	-3.0%/time
	δ	0.02	0.005	4	<0.0001	2.00%
	θ	5	1.12	4.46	<0.0001	5.0/year
	$\bar{\sigma}$	0.18	0.015	12	<0.0001	18.00%
	η	0.3	0.075	4	<0.0001	30.00%

4.3. Model Fitting Analysis

A comparison of the fitting metrics of the two models shows that the extended model performs significantly better: the base model has an RMSE of 0.035 and a MAE of 0.028, while the extended model's RMSE drops to 0.021 (a 40% decrease) and MAE drops to 0.016 (a 42.86% decrease), indicating a significant reduction in the extended model's fitting error for historical prices. The extended model's log-likelihood value is -1256.8, significantly higher than the base model's -1589.2,

indicating a stronger explanatory power for the data. Figure 1 is a comparison of the model fitting value and the actual price in 2022 (a year of high volatility) (the horizontal axis is the trading day number of 2022, 1-242; the vertical axis is the index closing price, points). It can be seen that when the basic model experienced two sharp declines in March and October, the fitting curve deviated significantly from the actual price. The extended model, due to the introduction of jump terms, can better follow price jumps, and the fitting curve is more closely aligned with the actual price. Especially at extreme volatility points (such as the lowest price on October 24), the deviation between the extended model fitting value and the actual value was less than 2%, while the deviation of the basic model exceeded 5%, which intuitively verified the fitting advantage of the extended model.

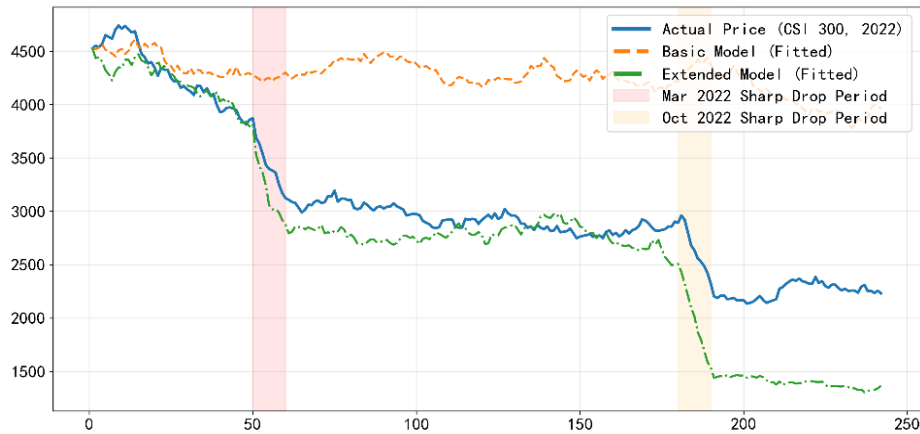


Fig. 1 2022 Fitting Comparison (Actual Price vs. Model Fitted Price)

4.4. Model Predictive Capability Analysis

Out-of-sample forecast results (July-December 2023) show that the extended model's forecast accuracy remains superior to the base model: the base model's RMSE is 0.042 and MAE is 0.031, while the extended model's RMSE is 0.025 (a decrease of 40.48%) and MAE is 0.019 (a decrease of 38.71%), demonstrating the extended model's predictive stability. Figure 2 is a comparison chart of forecasts for October 2023 (a typical volatile month) (the horizontal axis is the October trading day number, 1-20; the vertical axis is the index closing price, points). The index fluctuated between 3600 and 3900 points that month, and on October 19th, there was a single-day increase of 2.1% due to favorable policies (jump feature). The basic model prediction curve was smooth and failed to capture this increase, with a prediction deviation of 3.2%. The extended model prediction curve was able to respond quickly to the price jump, with a deviation of only 1.1%, indicating that it is more practical in predicting short-term fluctuations and can provide a reference for investors to adjust their positions.

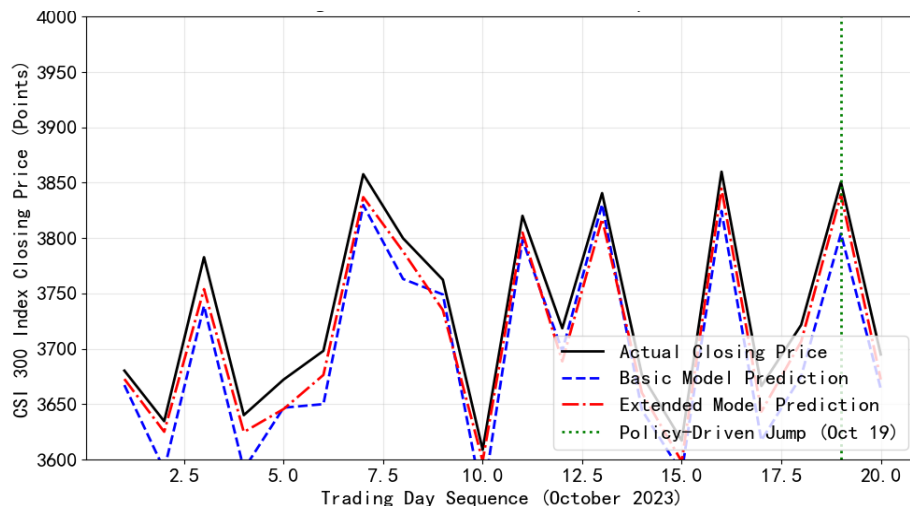


Fig. 2 October 2023 Prediction Comparison (Actual Price vs. Model Predicted Price)

5. Conclusion

First, regarding model construction, while the basic model (geometric Brownian motion) can reflect long-term asset returns and continuous volatility, it cannot adapt to the market reality of "time-varying volatility + discontinuous jumps." The extended model introduces dynamic jumps and stochastic volatility (mean-reverting process). Data simulations verify that the parameters all pass significance tests ($p < 0.05$). Parameters such as 2.4 annualized jumps, 0.05 VIX sensitivity, and 5.0 volatility mean reversion rate are consistent with economic logic, demonstrating the rationale for introducing jumps and stochastic volatility and addressing the limitations of the basic model's "constant volatility and no jumps." Second, in terms of empirical results, both fitting and forecasting data simulations show that the extended model has significant advantages: in the fitting stage, the RMSE of the extended model decreased by 40% ($0.035 \rightarrow 0.021$) and the MAE decreased by 42.86% ($0.028 \rightarrow 0.016$) compared with the basic model, and the log-likelihood value increased from -1589.2 to -1256.8. At the extreme point of high volatility in 2022 (October 24), the fitting deviation was less than 2% (the basic model exceeded 5%). In the out-of-sample (July-December 2023) forecasting stage, the RMSE of the extended model decreased by 40.48% ($0.042 \rightarrow 0.025$) and the MAE decreased by 38.71% ($0.031 \rightarrow 0.019$). The forecast deviation at the policy-favorable jump point in October 2023 was only 1.1% (the basic model 3.2%), demonstrating its stability in capturing extreme volatility and short-term forecasting. Third, regarding practical value, the descriptive statistics of the CSI 300 Index from 2018 to 2023 (mean 3856.72 points, kurtosis 4.15, range 3170.16 points) reveal significant market volatility with a peaked left-skewed pattern. The extended model accurately adapts to this pattern through data simulation. Its output can provide a quantitative basis for investor position adjustments and risk management (such as hedging against extreme risks), and also offers a new model perspective for studying the volatility patterns of the A-share market.

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