

Quantitative Pricing of Complex Convertible Bonds Based on The Geometric Brownian Motion Model Via Monte Carlo Simulation

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Abstract. Convertible bonds are crucial derivatives in quantitative finance. Their complex terms like automatic redemption and barrier prices, coupled with stocks' dynamic pricing features, pose precise pricing challenges. Traditional analytical methods struggle to capture the interactions among multiple factors. Thus, this study develops a quantitative pricing framework for complex convertible bonds by integrating the geometric Brownian motion (GBM) model with Monte Carlo simulation. Using Tesla's closing prices from January 6, 2020, to January 11, 2022, as the sample, this article estimated the GBM parameters (drift rate is 0.4428 and volatility is 1.2866) via log returns. Finally, this paper simulated 10,000 future one-year stock price paths and retained 4,895 valid paths after normal distribution filtering. Based on risk-neutral pricing theory, this paper priced a 13.25% p.a. USD Autocallable Barrier Reverse Convertible issued by Credit Suisse London Branch (face value USD 5,000). The theoretical price was USD 3,593.87, and repeated simulations show the price stabilizes above USD 3,500. Risk analysis reveals that 72.54% of valid paths indicate a stock price decline after one year, and 37.4% of valid paths trigger barrier clauses (consistent with 2022-2023 actual trends), validating the framework's feasibility.

Keywords: Convertible Bonds, Geometric Brownian Motion (GBM), Monte Carlo Simulation, Quantitative Pricing.

1. Introduction

In quantitative finance, convertible bonds combine the characteristics of both bonds and stocks. Their valuation requires integrating factors such as the dynamic pricing of the underlying stock, embedded clauses, and credit risk. Traditional analytical methods struggle to accurately capture the interactions among these factors. Monte Carlo simulation, known for its flexibility in handling path-dependent features and adaptability to stochastic processes, has become a critical tool for solving such pricing challenges. In pricing convertible bonds in China, Monte Carlo simulation combined with default intensity models can effectively capture the coupling relationship between credit risk and complex clauses [1]. Geometric Brownian Motion (GBM), as a core model for asset price dynamics, assumes independence and normality of price increments, making it a foundational theory for asset pricing in quantitative finance [2].

The core assumptions of GBM show clear mismatches with actual financial market behaviors. For example, the model's assumption of independent increments deviates from the long memory observed in markets, driving academic efforts to find more realistic alternatives. Fractional Brownian motion (fBm), which uses the Hurst index to capture price correlations, theoretically improves upon GBM but faces empirical limitations, such as in the Athens Stock Exchange [3, 4]. Some studies also enhance fBm's predictive accuracy for asset price trends using machine learning algorithms [5].

Extensions of GBM, such as the reflected geometric Brownian motion (RGBM), are controversial. The reflecting boundary in RGBM may introduce arbitrage opportunities and break classical boundary conditions in derivative pricing formulas [6]. Meanwhile, research on the asymptotic behavior of the Laplace transform of GBM time integrals provides theoretical support for pricing fixed-income products like zero-coupon bonds. However, its applicability differences across time scales need deeper analysis [7].

To address the above issues, academia has proposed multiple optimization directions: introducing state transition mechanisms to enable models to adapt to characteristic differences of markets in different phases (e.g., bull and bear markets) [8]. optimizing models' fitting of non-normally distributed data via entropy correction methods to cope with the leptokurtic and fat-tailed characteristics often exhibited by asset returns [9]. Meanwhile, the time-scale characteristics of underlying stocks are equally non-negligible in convertible bond pricing practice, as data with different stock period lengths significantly affect pricing results [10].

Although academia has optimized traditional models through various methods, the pricing of complex convertible bonds—embedded with provisions such as automatic call and barrier price—still requires more accurate capture of the interaction between the price dynamics of underlying assets and clause trigger mechanisms. Based on this, this study aims to combine Monte Carlo simulation with geometric Brownian motion to solve the pricing problem of complex convertible bonds and provide a more rational decision-making basis for investors.

The next section, this paper will briefly introduce the theoretical knowledge of convertible bonds. Then, it will select specific convertible bonds and analyzes the system of quantitative modeling methods in this study. The following part will present the pricing results and risk analysis process. At last, this essay will elaborate on the feasibility and limitations of the GBM-Monte Carlo framework in pricing complex convertible bonds, and further proposes directions for subsequent model improvements.

2. Theoretical Knowledge

2.1. Bond Attribute

Similar to ordinary corporate bonds, convertible bonds explicitly stipulate the coupon rate, maturity period, and principal redemption rules. During the holding period, investors can receive fixed interest income regularly. If no special clauses are triggered, investors may require the issuer to redeem the principal plus interest at par value upon maturity. Even if the price of the underlying stock plummets, this attribute provides investors with a minimum return guarantee.

2.2. Equity Attribute

Within the agreed convertible period, investors have the right to convert their held convertible bonds into the issuer's common stock at the predefined conversion ratio. When the market price of the underlying stock exceeds the conversion price, investors can obtain spread gains by selling the converted stocks, thereby sharing the returns from the appreciation of the company's equity. If the stock price is lower than the conversion price, investors may abandon conversion and continue to hold the bonds to ensure the safety of principal and interest.

2.3. Example Parameters

This case study focuses on a convertible bond issued by a Chinese listed company. The bond has a face value of 100 yuan, a 2.0% annual coupon rate, and a 5-year maturity. Each bondholder receives 2 yuan in interest annually (calculated as $100 \times 2\%$). For the conversion clause, the initial stock price is 20 yuan per share, and each bond can be converted into 5 shares of the underlying stock. The redemption clause allows the issuer to redeem the bond at 103 yuan per bond if the stock price stays above or at 130% of the initial price (26 yuan per share) for 20 consecutive trading days. Conversely, the put clause enables investors to sell the bond back to the issuer at 101 yuan per bond if the stock price remains below or at 70% of the initial price (14 yuan per share) for 30 consecutive trading days.

3. Method

The GBM model used in this study is a stochastic process model widely used in financial derivative pricing. It assumes that the logarithmic changes of asset prices follow Brownian motion, ensuring

asset prices remain positive, in line with real-world market characteristics. The core of the GBM model lies in its stochastic differential equation, expressed as:

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad (1)$$

where $W(t)$ is a Wiener process, μ is the percentage drift, and σ is the percentage volatility (constant). By Ito's lemma:

$$d \ln S(t) = \underbrace{\left(\mu - \frac{\sigma^2}{2} \right)}_v dt + \sigma dW(t). \quad (2)$$

For any initial value $S(0)$, its analytical solution is:

$$S(t) = S(0) \exp \left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \right). \quad (3)$$

In this study, by collecting a set of historical data series of Tesla's stock closing prices: $S(t_0), S(t_1), \dots, S(t_N)$, and since:

$$\ln S(t + \Delta t) - \ln S(t) = v \Delta t + (\sigma W(t + \Delta t) - W(t)), \quad (4)$$

the parameter estimates are obtained as:

$$\hat{v} = \frac{\bar{r}}{\Delta t} = \frac{\sum_{j=1}^N r(t_j)}{N \Delta t}, \quad \hat{\sigma}^2 = \frac{\sum_{j=1}^N (r(t_j) - \bar{r})^2}{(N - 1) \Delta t}. \quad (5)$$

The Monte Carlo simulation method is employed to generate a large number of asset price paths, with each path representing a possible evolutionary scenario of the underlying stock price. Based on Risk-Neutral Pricing Theory, in an arbitrage - free market, the current derivative price equals the expected value of future cash flows under the risk - neutral probability measure, discounted at the risk - free rate to the present [11]. This theory assumes all market participants are risk - neutral, meaning they only require a risk - free rate of return and not extra compensation for risk - bearing. Thus, the risk - related value of derivative future payoffs need not be separately priced; instead, pricing can be based on the average of all possible payoffs (expected payoffs).

$$V = \frac{1}{N} \sum_{i=1}^{n=N} \text{payoff} \times e^{-rt_i}. \quad (6)$$

The risk - free rate was obtained from Investing.com, based on the US one - year Treasury rate on the derivative's issuance day, at 0.4425%. All model computations were performed in R Studio.

4. Results and Discussion

4.1. Modeling Estimation

This study selects the 1-year 13.25% p.a. USD Autocallable Barrier Reverse Convertible issued by Credit Suisse AG, London Branch, which is linked to Tesla, a well-known U.S. stock. The selected Tesla stock data are sourced from Yahoo Finance. Considering that the Initial Fixing Date and Final Fixing Date of this convertible bond are January 12, 2022, and January 17, 2023, respectively, the sample data covers trading days from January 6, 2020, to January 11, 2022.

An analysis of Tesla's stock data reveals that the stock was listed in June 2010 and has exhibited high volatility over the past five years. Data from the early years after its listing fails to reflect the stock's characteristics; furthermore, ignoring inflation would cause significant errors in future data estimation. Thus, the selected two-year historical data are sufficient to reflect the stock's recent trends. Figure 1 presents the initial time series chart of the stock, plotted using the opening price, closing

price, highest price, and lowest price from 2020 to 2021, with the abscissa representing trading dates and the ordinate representing stock value (Unit: USD).



Fig. 1 Initial stock data time-series plot

Pricing modeling is conducted for the research object selected in this paper, and Table 1 presents the information of necessary terms involved in the pricing of this convertible bond.

Table 1. 13.25% p.a. USD Autocallable Barrier Reverse Convertible

Key fixed terms	Content	Key fixed terms	Content
Reference Share	Tesla Inc.	Coupon	13.25% p.a., paid quarterly
Initial Level	USD1106.22	Barrier (50%)	USD553.11
Initial Fixing Date	2022/1/12	Final Fixing Date	2023/1/17
Early Redemption Observation Date	quarterly, for the first time after 6 months	Early Redemption Level (100%)	USD 1,106.22
Conversion Ratio	4.5199	Currency/Denomination	5000USD

4.2. Monte Carlo Simulation

Calculations via R.Studio generated 10,000 simulation paths, yielding the final results as shown in Table 2. In processing the simulation path data, extreme outliers arising from the limitations of the GBM model are irrelevant to subsequent result analysis and conclusion drawing. Therefore, the normal distribution is used to remove such outliers. This paper set $\mu(\text{year} - \log\text{return})$ is 0.4428 and $\sigma(\text{year} - \text{variance})$ is 1.2866, and put them into the formula (7). Finally, this paper gets a calculated rightmost boundary (maximum stock price) of 3192 for the selected sample.

Table 2. The estimation results of derivatives payoffs

Scenario	Results
Barrier not touched	0
Barrier touched	0.2338
Early redemption	0.7662
Derivative price	3593.87USD

$$F(x) = \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt. \quad (7)$$

Figure 2 below shows the simulated trends of Tesla stock one year later. The blue line on the left represents the historical path, and the colored lines on the right represent the simulated paths, with the abscissa being trading days and the ordinate being stock price (Unit: USD). Removing extreme data, 4895 valid paths were finally retained, the pricing of this derivative is 3593.87 USD, with the mean closing price of the sample stocks being 953.43 USD and the median being 785.39 USD. After adjusting the value of the random seed multiple times, it is consistently found that the pricing exceeds 3500 USD.

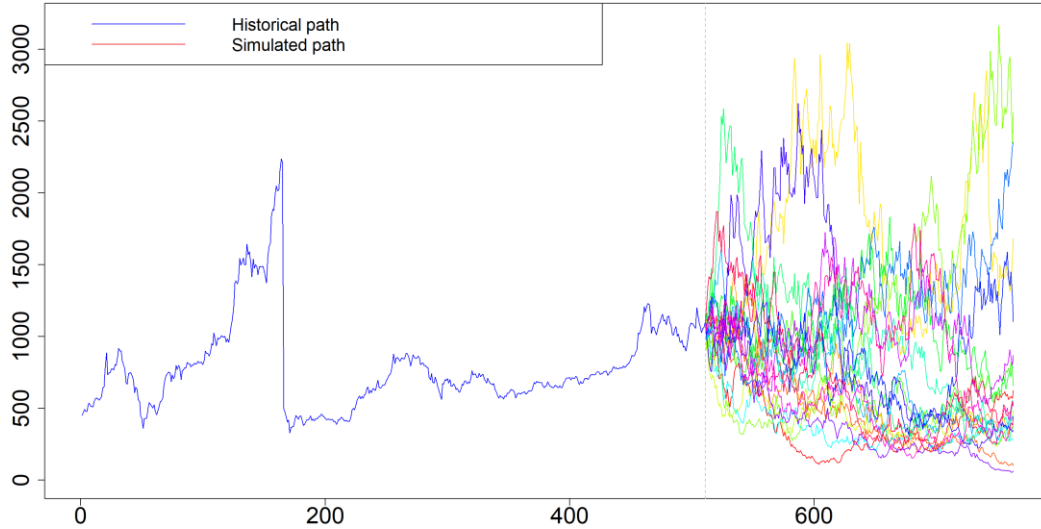


Fig. 2 Simulated trends of Tesla's stock closing prices over the next year

4.3. Risk Analysis Based on Data

Since investors can't predict the future stock trend over one year, pricing helps to reasonably infer the loss scenarios when the future stock price fluctuates within a fixed range. Assuming the convertible bond price is 3,500 USD and substituting into equation (8):

$$\text{couponpayment} * 4 + k * \text{Demon} \leq 3500\text{USD}. \quad (8)$$

Then the value of k can be calculated and the result is 0.5675. The barrier clause implies that losses are certain when the stock price drops below 50% of the initial price. This paper needs to calculate the proportion of paths falling below the barrier price. Table 3 shows the distribution results.

Table 3. The distribution of stock closing prices below the initial price

Price	Number	Percentage
$\text{Price} \leq 553.11$	1831	37.4%
$553.11 < \text{Price} \leq 1106.22$	1720	35.1%

Figure 3 compares actual and simulated stock price trends. The x - axis shows trading days, and the y - axis shows stock prices in USD. The dark green line is the actual stock trend, the black long - dashed line is the initial price (1,106.22 USD), the blue dash - dotted line is the barrier price (553.11 USD), and the red short - dashed line is the actual final price (122.4 USD). It shows that most paths have year - end closing prices below the initial price, with many falling below the barrier price.

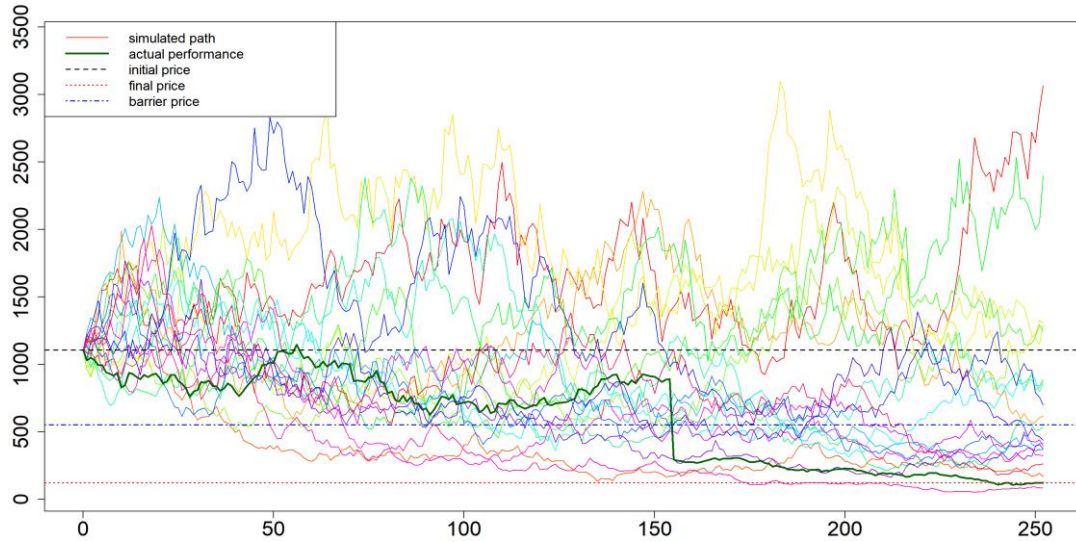


Fig. 3 Comparison of actual and simulated paths of Tesla stock prices

From the 4,895 valid samples, 72.54% showed a year - on - year drop in the stock's closing price, and 37.4% of paths indicated losses for investors holding the convertible bond. Based on the GBM model and Monte Carlo simulation, risk-averse investors would not buy Tesla stocks or this bond.

To evaluate the model's fit, Figure 4 uses the Mean Squared Error (MSE) principle to compare deviations and shows the optimal simulated path against the actual stock trend. The black long - dashed, blue dash - dotted, and red short - dashed lines have the same meanings as in Figure 3. The dark red line is the optimal simulated path, and the dark green line is the actual stock trend.

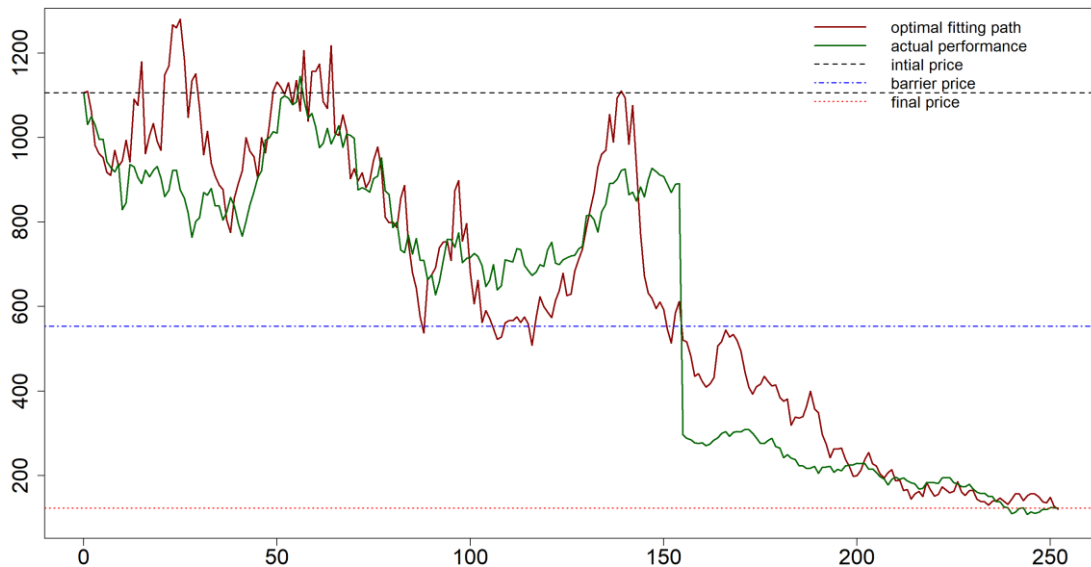


Fig. 4 Comparison of optimal fitting path and real stock price trend

The relative errors of the optimal path were analyzed by counting each trading day of the simulation. Table 4 shows the simulation relative errors at various key time nodes, including the overall average relative error, the redeemable window, and the barrier price trigger points. The overall error was 19.99%. Given the Tesla stock's annualized volatility of 1.28 (much higher than the 20% - 30% of ordinary stocks), the fit is relatively good, capturing the stock price's essential trend and not affecting the loss - triggering judgment of 37.4%. At the second redeemable window, the relative error was 61.55%. Although high, the simulated price was much lower than the redeemable price, avoiding misjudgments in pricing decisions. At the final trading day, the relative error was only 2.25%, indicating the simulated closing price's deviation was small and trending similarly to the actual price. This supports the conclusion that the convertible bond's theoretical price can stably exceed 3,500 USD.

Table 4. Relative errors at different time nodes

Date	Relative errors
2022/7/15	12.55%
2022/10/13	61.55%
2023/1/13	2.25%
Mean	19.99%

5. Conclusion

This study uses Tesla stock data (2020-2022) with the GBM model to estimate drift, volatility, and Monte Carlo simulation generates numerous price paths. The convertible bond (with a 5000-USD face value) is theoretically priced at 3593.87 USD, and repeated simulations show a stable price above 3500 USD. After excluding extremes via normal distribution, 72.54% of simulation paths show a stock price drop after one year, and in 37.4% of cases, the price falls below the barrier level, causing investor losses. This aligns with the actual downward trend of Tesla's stock price during the bond's term and the loss - risk when the price breaks the barrier. The combined GBM model and Monte Carlo simulation offer a workable framework for convertible bond pricing. Given the complexity of this derivative, including redemption and barrier clauses and interest - payment methods, this approach solves its pricing problem and can be applied to other pricing issues in quantitative finance.

The GBM model's assumptions of independent stock price increments and constant volatility limit its ability to predict Tesla's pronounced market trends (like the 2022 - 2023 drop) and its fluctuating volatility. It also has flaws in extreme market conditions, such as Tesla's charging station incident - induced stock plunge. These limitations cause deviations between simulated and actual paths, making many investors fail to anticipate Tesla's stock devaluation and subsequent losses. This may be why the derivative wasn't reissued.

These findings suggest that for financial institutions involved in derivative design and pricing. It is necessary to introduce more flexible models to enhance pricing accuracy when dealing with complex convertible bonds linked to highly volatile underlying assets in the future. The future work could attempt to incorporate occupation time techniques with Markov chains to derive models that overcome the previous limitation of only handling specific payoff functions. Alternatively, applying Financial Time Series Generative Adversarial Networks (FinGAN) to pricing may more accurately reproduce stock return processes under risk neutrality. At last, incorporating the impact of macroeconomic policy fluctuations on dynamic asset prices into research represents another improvement direction to enhance pricing precision in the future.

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