Evaluating an Enhanced Monte Carlo Framework for Bull Contract Valuation and Risk Assessment: A GARCH-T Empirical Study

Chengxi Luo [1, *], Yutong Liu [2] and Shuodong Li [3]

[1] Department of Mathematical Sciences, Beijing Normal-Hong Kong Baptist University, Zhuhai, China

[2] School of Mathematics and Artificial Intelligence, Chongqing University of Arts and Sciences, Chongqing, China

[3] School of Finance and Investment, Guangdong University of Finance, Guangzhou, China

* Corresponding Author Email: s230018051@mail.uic.edu.cn

Abstract. Traditional Monte Carlo simulations have been widely used in the field of financial derivatives pricing owing to their capacity in handling complex path dependent structures. However, they are limited in their ability to capture volatility clustering and fat-tailed distributions in financial markets. This study proposes an enhanced approach to assist in pricing derivative warrants with mandatory redemption features by integrating GARCH models and t-distribution assumptions into the Monte Carlo framework. To validate its superiority in capturing volatility clustering and fat-tailed phenomena, the results are compared against those obtained from conventional Monte Carlo simulations. Empirical results indicate that this method improves predictive accuracy and risk measurement, providing a more precise depiction of price dynamics and serving as a more reliable tool for derivative valuation and risk management. The framework contributes to the broader field of computational finance by offering a robust tool for the valuation of complex derivative products and serves as a foundational reference for future research on structured products in emerging markets.

Keywords: Monte Carlo Simulation, GARCH Model, t-distribution, Structured Products, Risk Measurement.

1. Introduction

Financial derivatives, as a fundamental component of the modern financial system, have their development depth and breadth serving as critical indicators of market maturity. Through functions such as risk transfer, price discovery, and market completeness, they significantly enhance capital allocation efficiency and risk management capabilities. As the complexity and uncertainty of financial markets continue to escalate, the valuation and pricing of financial derivatives have become focal points of academic and industry research. Complex structured products, such as callable bonds and barrier options, which rely on the underlying asset's price path and exhibit pronounced tail risks, impose stringent requirements on pricing models' accuracy and their ability to capture extreme risk events. Traditional methods such as the Black-Scholes model rely on stringent assumptions—such as constant volatility and frictionless markets—that tend to underestimate risk parameters like Value at Risk (VaR) during periods of extreme market events. In addition, it just focusses solely on the asset's terminal price, resulting in systematic deviations between theoretical valuations and actual market prices, thereby compromising pricing accuracy and limiting the practical applicability of their self [1]. In contrast, the Monte Carlo simulation method focuses on the entire price trajectory and can approximate the true theoretical value of assets through extensive simulations, thus serving as the cornerstone for pricing path-dependent derivatives [2]. However, current research still faces significant challenges in simulation optimization. Consequently, improving the valuation framework of Monte Carlo simulations to more accurately capture market volatility characteristics and establish a more reliable foundation for precise pricing remains a critical and unresolved issue.

This study aims to address the problem which leads to the aforementioned pricing biases at the fundamental level of theoretical valuation. Conventional Monte Carlo simulations typically assume constant volatility and normally distributed returns, neglecting common features of financial time series such as volatility clustering and fat tails. To this end, the primary objective of this study is to build an enhanced Monte Carlo framework that integrates GARCH family volatility models with heavy-tailed distributions. Empirical validation is conducted using Alibaba's bull certificates on the Hong Kong market, explore the superiority of this improved framework over traditional methods in terms of value measurement accuracy and risk management. The research seeks to explore more flexible option pricing models, optimize the valuation of path-dependent options, and provide market participants with more precise theoretical guidance and practical insights.

The subsequent sections of this paper will revolve around these four parts: the literature review, research methodology, empirical analysis, and conclusions with prospects. The literature review will systematically examine existing research on asset price simulation, the GARCH family models, and their applications in financial valuation. The methodology section will detail the design framework of traditional Monte Carlo simulation and the enhanced GARCH-Monte Carlo simulation incorporating improved distribution assumptions, along with an overview of the experimental data and evaluation metrics. Finally, the empirical analysis will present and compare the performance of various simulation paths in terms of key indicators and valuation accuracy.

2. Literature Review

2.1. Application of Monte Carlo Methods

The application of Monte Carlo simulation in derivatives pricing has become increasingly prevalent, highlighting its unique value. Some studies have verified the superiority of Monte Carlo simulation during periods of financial market turbulence, noting that its predictive accuracy is significantly higher than traditional methods under special market conditions [3]. Other research has utilized Monte Carlo simulation to analyze the impact of specific events (such as the COVID-19 pandemic) on asset price volatility, confirming its unique capability in capturing extreme market events [4]. More cutting-edge works have employed Monte Carlo simulation to provide new testing methods for the efficient market hypothesis [1]. These studies indicate that this method can flexibly handle various complex payoff structures and market conditions when addressing the valuation and pricing of complex derivatives (such as path-dependent options and structured products), offering robust support for financial innovation.

2.2. Challenges and Improvements of Monte Carlo Methods

While the Monte Carlo method demonstrates significant advantages in mitigating the curse of dimensionality, it still frequently faces challenges related to computational time consumption [5]. Some researches focused on the pricing of equity-indexed annuities, introducing a scenario matrix technique within the G2++ two-factor model to streamline the Monte Carlo simulation, thereby achieving substantial gains in computational efficiency [6]. Furthermore, some studies have optimized the input parameters of Monte Carlo simulations by introducing machine learning prediction models (linear regression), improving the accuracy and efficiency of yield predictions and enhancing the effectiveness of simulated investment portfolios [7].

What's more, traditional Monte Carlo simulations, which rely on assumptions of constant volatility and normal distribution, often struggle to capture the real-world characteristics of financial markets, leading to model bias. One line of research addresses this by integrating jump-diffusion models with Monte Carlo simulation and incorporating importance sampling techniques. This approach effectively reduces valuation errors and offers a more accurate pricing solution for barrier options [8].

3. Research Methodology and Model Construction

3.1. Data Description and Preprocessing

3.1.1 Data sources and description

This study selects Alibaba Group Holding Limited (Stock Code: 9988) listed on the Hong Kong Exchange as the underlying asset. Daily closing price data from December 4, 2023, to July 11, 2025, was collected from the official website of the Hong Kong Exchange and the Lixinger Financial Data Platform. The corresponding structured product is a mandatory callable bull certificate (Code: 62584). Its daily closing prices and trading volume throughout the entire life cycle were extracted, with data also sourced from the official Hong Kong Exchange trading system. Additional product information is presented in Table 1.

Table 1. Core parameters of the bull certificate

Parameter name	Value	Unit	Comment
Exercise price	68.50	HKD	Permanently below the knock-out price
Redemption price	70.0	HKD	Trigger conditions for forced redemption
Exchange ratio	100:1	/	100 warrants per share
Cost of acquisition	0.25	HKD	/

According to the income mechanism of this structured product, its income function formula is:

$$Return = \ (S_T - K/R) \ - C. \tag{1}$$

Among these, S_T represents the settlement price (the official closing price of the underlying stock on the expiration date or at the time of mandatory call), K denotes the strike price, R refers to the conversion ratio, C stands for the initial cost. If the settlement price is lower than or equal to the strike price ($S_T \leq K$), the investor will lose the entire initial investment. In subsequent research, this payoff function will be applied to evaluate the value of the product based on asset price paths generated through the dual-channel simulation method.

3.1.2 Data preprocessing

To satisfy the additivity of the return sequence and avoid the problem of negative price, so that it conforms to the assumption of geometric Brownian motion (GBM), this study calculates the logarithmic return rate of the original price data:

$$r_t = ln\left(\frac{P_t}{P_{t-1}}\right), \tag{2}$$

where P_t denotes the asset price in the current period, P_{t-1} represents the asset price from the previous period. In this study, this formula is used to calculate the daily logarithmic returns of the original asset price path, with the results being applied to subsequent Monte Carlo simulations. As the result of the limited sample size (N=252) and the research objective involving modeling fat-tailed effects, the analysis retains the original data's distribution characteristics without applying tail trimming to outliers [9].

3.2. Selection and Explanation of Indicators

VaR, the maximum potential loss that a specific financial asset or portfolio may incur over a designated future holding period, with the given fixed confidence level. In its calculation, α is typically set at 95% or 99%, with the formula expressed as:

$$VaR_{\alpha}\,(x) = -\min\{\text{x} \mid P(r \leq x) \geq \alpha\} \quad (0 < \alpha < 1) \tag{3}$$

Expected Shortfall (ES), also known as Conditional Value at Risk (CvaR) or Average Value at Risk (AVaR), represents the average of tail losses beyond the VaR at a specified confidence level under extreme loss scenarios. It is employed to quantify the average magnitude of tail risk. Its mathematical formulation is as follows:

$$ES_{\alpha}(x) = -\mathrm{E}[\, x \mid x \geq -VaR_{\alpha}(x)\} \quad (0 < \alpha < 1) \tag{4}$$

Root Mean Square Error (RMSE), calculated as the square root of the mean of the squared deviations between the model's predicted prices and the actual market prices, serves as a comprehensive metric for quantifying the overall accuracy of price forecasts. A lower RMSE indicates superior predictive performance. Its formula is expressed as:

$$RMSE = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(\widehat{p_t} - p_t)^2} \tag{5}$$

Kurtosis is a statistical measure that describes the peak of a data distribution and the heaviness of its tails. A normal distribution has a kurtosis of 3. Therefore, positive excess kurtosis indicates "fat tails", suggesting the probability of extreme values occurring is significantly higher than that predicted by a normal distribution. The return rates of financial assets often exhibit fat-tailed characteristics, and kurtosis helps serve as a measure to assess whether the distribution of asset returns deviates from the normality assumption. Its formula is:

$$Kurt\,(x) = \frac{E[(X-\mu)^4]}{\sigma^4} \tag{6}$$

In this study, variables $VaR_{95\%}$ and $\mathrm{ES}_{95\%}$ are selected as risk measurement indicators to quantify the extreme downside risk faced by structured products within their underlying asset's simulated price paths, thereby facilitating the assessment of the efficacy of various simulation methods in capturing extreme losses. RMSE is employed to compare the valuation accuracy of different Monte Carlo simulation methods for the bull callable bearish contracts numbered 62584, assessing the model's fit to market prices. The calculation of kurtosis provides a basis for assessing the fit of simulated distributions, such as the t-distribution and GED, to empirical data.

3.3. Method Introduction

This research first verified the stability of the return series using the ADF test to ensure that it meets the GARCH modeling requirements, and the test results show that the return series is stationary. To evaluate the effectiveness of the improved method, this study designed a comparative experiment between the traditional group and the improved group: the traditional method simulated the price path of the underlying asset from December 4, 2023, to July 11, 2024 based on the assumption of a normal distribution, used the last observed price and the historical volatility calculated from the period December 3, 2023, to December 3, 2024. In contrast, the enhanced approach first employed Q-Q plot analysis and ARCH-LM tests to verify fat-tailed distributions and volatility clustering, then applied a t-distribution coupled with a GARCH (1,1) model to simulate dynamic volatility. Tens of thousands of paths were simulated by both groups, and corresponding product values were computed. To assess the enhanced model's predictive accuracy and risk measurement capability, a comparison was conducted between the two groups based on risk metrics (VaR and ES), RMSE, forced redemption probability, and simulated price paths. The specific process is illustrated in Figure 1.

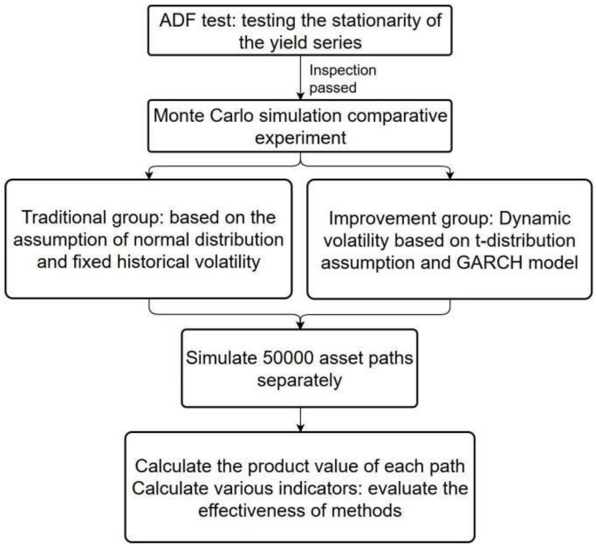

Fig. 1 Comparative experimental workflow

3.4. Model Testing

The study first conducted ADF stationarity test on the return series. As shown in Table 2, the test statistic value was -14.8119 (p-value < 0.0001), which is significantly lower than the 1% critical value at the zero-lag level. At the 1% significance level, the null hypothesis of the existence of unit roots is rejected, indicating that the series is stationary.

Table 2. ADF test result

Statistic	P-value	Lags	1% Critical Value	5% Critical Value	10% Critical Value
-14.8119*	0.0000	0	-3.4574	-2.8735	-2.5731

* The null hypothesis of non-stationarity is rejected at the 1% significance level

Based on the stationary property of the return series, this study further utilized the Q-Q chart to perform tail testing on it. Figure 2 illustrates the Q-Q chart of the return series, it shows that the data points are mostly centered around the reference line in the middle region but deviate significantly from the reference line in the tails, forming a typical “S” shape. This indicates that the tail thickness of the actual yield rate distribution is significantly greater than that of a normal distribution, suggesting the presence of a “tail” phenomenon, where extreme returns occur with a higher frequency than predicted by a normal distribution.

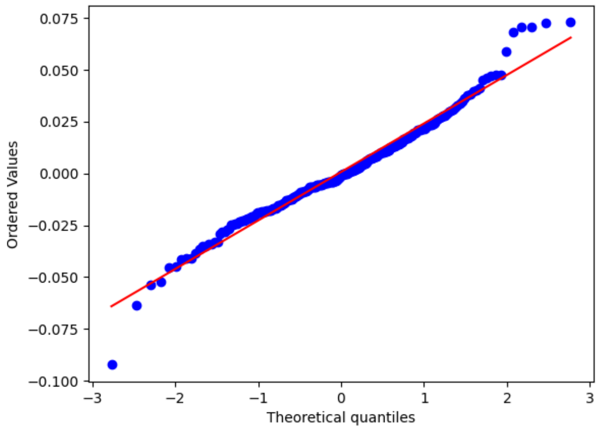

Fig. 2 Q-Q Plot for fat-tailedness test

To validate the presence of volatility clustering, this study conducted an ARCH-LM test on the mean equation residuals. The test results are as shown in Table 3, with all statistical significance values less than 0.01, indicating that the null hypothesis of no ARCH effect can be rejected at the 1%

significance level, suggesting that the residual series exhibits significant volatility clustering. This provides empirical support for the use of GARCH models.

Table 3. ARCH-LM test result

Test Item	Statistical Value	p-value
LM Test	23.5091	0.0003
F Test	5.0821	0.0002

3.5. Model Selection and Fitting Results

In the Monte Carlo simulation using the improved method, the study selected the GARCH (1,1) model to generate dynamic volatility paths, which is simple in structure but sufficient to capture the most critical volatility clustering and persistence in financial time series. With a faster computing speed, the simulation ensures numerical stability and interpretability while avoiding overfitting. The ADF test used in preceding work with zero-order lag terms indicates that the return series exhibits no significant autocorrelation, leading the research to adopt a mean equation with only a constant term:

$$r_t = \mu + \varepsilon_t \tag{7}$$

The volatility equation is:

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2 \tag{8}$$

After a series of rigorous model fitting and validation procedures, the final estimated values of the model parameters are as follows:

Table 4. GARCH (1,1) model parameters

Parameters	Estimation	Standard error	t-value	p-value
$\boldsymbol{\omega}$	7.53e-05	1.71e-05	4.410	< 0.001
$\boldsymbol{\alpha_1}$	0.0787	0.0400	1.970	0.049
$\boldsymbol{\beta_1}$	0.7863	0.0400	19.684	< 0.001
$\boldsymbol{\nu}$	7.1446	2.765	2.584	0.010

As shown in Table 4, the estimates of all parameters are statistically significant at the 5% level ($p < 0.05$), with ω and β_1 being highly significant at the 0.1% level. The model exhibits high persistence ($\alpha + \beta = 0.865$) and the residual shows significant heavy tail features ($\nu = 7.14$). This model will be used in the Monte Carlo simulations for the improvement of the conditional volatility predictions.

4. Empirical Analysis and Result Discussion

4.1. Overview of Simulation Results

This study employs a traditional Monte Carlo simulation (TMC) assuming normality and static volatility, as well as an improved Monte Carlo simulation (IMC) incorporating t-distribution characteristics and GARCH (1,1) volatility modeling, to simulate the price paths of the underlying asset. Figure 2 provides a visual comparison of the generated asset price paths from both methods.

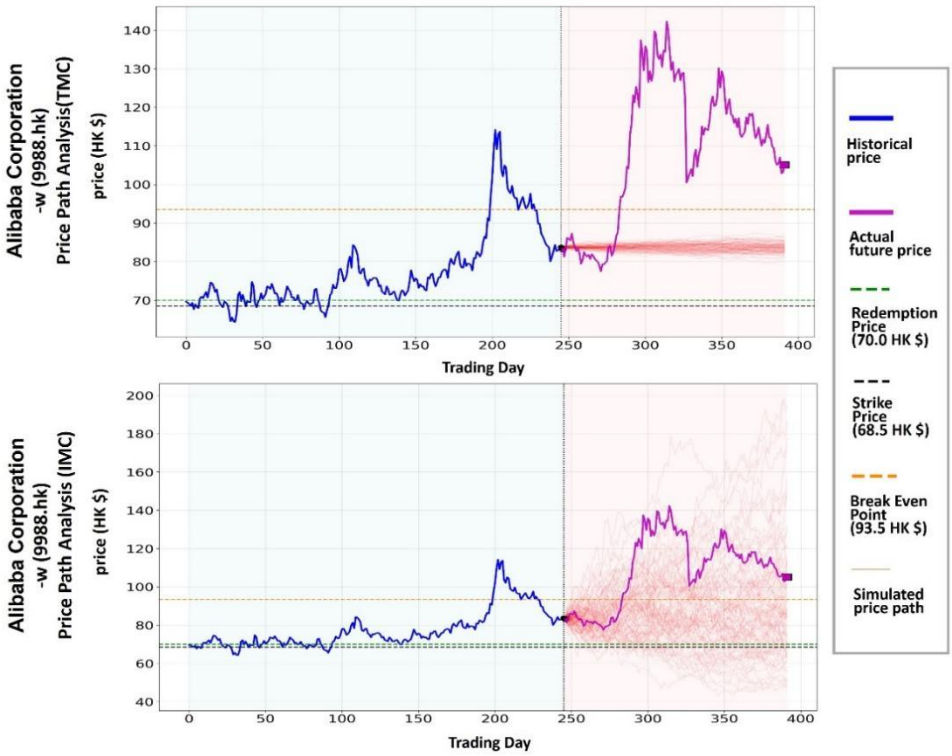

Fig. 3 Comparison of Simulated Asset Price Paths: Traditional vs. Improved Monte Carlo Methods

As can be seen from Figure 3, compared to the traditional Monte Carlo method, the improved method provides significantly greater coverage of the actual price data within the prediction interval, indicating a stronger predictive accuracy. This suggests that the improved Monte Carlo model more accurately captures price dynamics and trends [10].

4.2. Mandatory Redemption Situation

In this study, TMC method predicted the number of forced redemption events as zero, seriously underestimated the probability of its triggering, while the IMC method predicted a forced redemption proportion of 48.55% and its estimated triggering probability aligns more closely with the empirical probability. The reason lies in the fact that TMC method relied on traditional assumptions, failing to capture the dynamic changes in the volatility of the underlying assets, which made it difficult to reach the redemption level when the starting price level and volatility mean were both relatively low. In contrast, the IMC method, which incorporated a GARCH model and successfully captured the short-term increase in volatility during the preceding two months, leading to more accurate risk estimates. The model still demonstrated reasonable triggering probabilities even at lower starting price levels.

Figure 4 shows the forced redemption distribution across the ensemble of price paths simulated by the IMC group. As shown in the figure, the probability of extreme values in the settlement price decreases as the redemption time occurs later in the path. This is due to path dependency, which causes paths that did not trigger forced redemptions in the early periods to converge in the simulation, thereby reducing the occurrence of extreme paths. Additionally, market volatility exhibits an aggregation effect, redemption events occurring more frequently during volatility clustering periods, which means the subsequent reduction in volatility will lead to a lower probability of extreme values occurring. It is noteworthy that the β of this GARCH model is 0.78, indicating that the volatility is relatively persistent, which suggests that the model may have failed to accurately capture all volatility clustering effects.

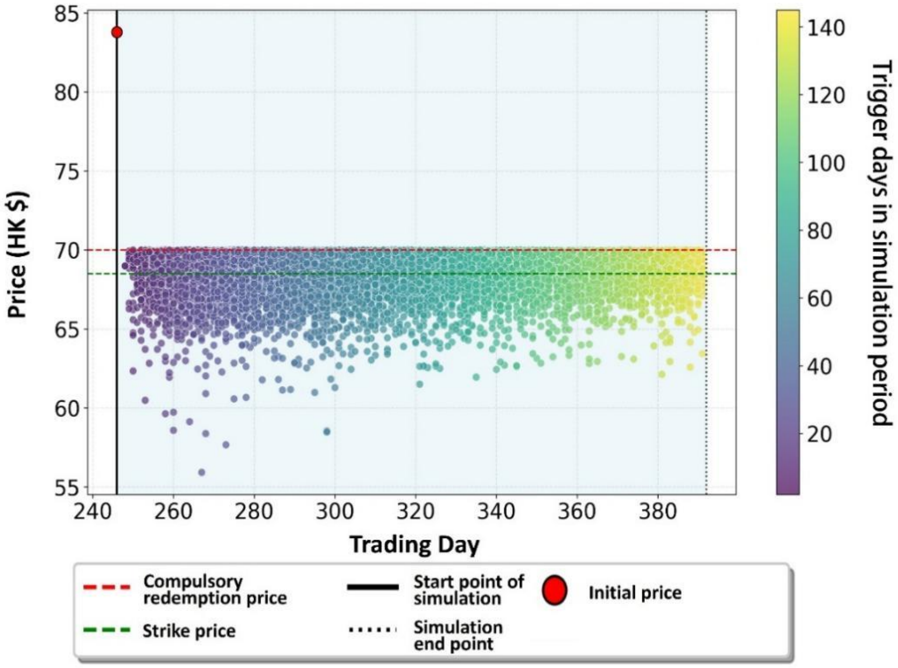

Fig. 4 Distribution of forced redemption events

4.3. Comparison of Prediction Accuracy

The Improved Monte Carlo Method significantly outperforms the Traditional Monte Carlo Method in all key accuracy dimensions.

Primarily, TMC predicted a loss rate of 100%, which clearly does not align with the real-world market ecosystem. This is just like its significant underestimation of the forced redemption probability, which is also due to the low starting value and the low mean volatility levels in historical price data, leading to limited volatility in traditional groups and making it difficult to reach the profit-loss balance point. By incorporating a dynamic volatility model, IMC predicted a loss rate of 68.94%, which more closely captures the actual product loss scenarios.

In addition, this study calculated the RMSE based on the non-forced redemption path in the asset price path of TMC group and the actual price path, the result was HKD 31.27, while the same indicator calculated by the IMC method decreased to HKD 24.74, which represents a relative improvement of 20.87%. This result indicates that IMC has higher overall accuracy in price path fitting.

In conclusion, IMC outperforms TMC in both tail event probability accuracy and path estimation error, confirming the effectiveness of the improved volatility modeling.

4.4. Comparative Analysis of Risk Measurement

At the 95% confidence level, there is a significant difference in the estimation of extreme risks between the two methods. TMC predicted a VaR of −0.121 Hong Kong dollars and an ES of −0.127 Hong Kong dollars, while IMC estimated the VaR as 0.25 Hong Kong dollars and ES as 0.25 Hong Kong dollars. This is the result of TMC method significantly underestimating the probability of forced redemption events occurring, further confirming the fact that traditional methods are overly conservative in estimating tail risk, while IMC more accurately captures the true loss distribution.

Furthermore, IMC method increased the forced redemption probability from 0% to 48.55%, indicating that it more accurately captures the early knockout features of the product. Overall, IMC significantly outperforms TMC in both risk measurement accuracy and capability to model key product characteristics.

5. Conclusion

In this study, the GARCH (1,1) model under a t-distribution framework is integrated into the Monte Carlo simulation. On one hand, the estimated parameters (α=0.079, β=0.78) effectively capture volatility clustering and high persistence. On the other hand, the heavy tails of the return distribution are characterized by a t-distribution with degrees of freedom ν = 7.14. This approach simultaneously addresses two major limitations of conventional simulations: the inability of constant volatility assumptions to accurately predict risk, and the inadequacy of the normal distribution in capturing fat-tailed behavior consistent with real financial markets.

Empirical results indicate that the enhanced Monte Carlo simulation yields substantial improvements in predicting asset prices: the probability of mandatory redemption event surges from 0% to 48.55%, the risk of losses decreases by 31.06%, and the average warrant return increases by 28.87%. These outcomes align more closely with the actual value profile of the product and real-world financial market behavior. As a result, this refined approach can offer financial institutions more forward-looking insights for capital reserve planning, real-time risk control, and investment strategy development.

However, the model remains reliant on historical parameter estimation, which limits its ability to capture policy shocks or black swan events, and makes it susceptible to parameter drift under structural breaks. Furthermore, the current framework does not explicitly incorporate a jump-diffusion process, resulting in insufficient capture of sudden price jumps. Additionally, computational efficiency remains a significant challenge in the current improved Monte Carlo pricing approach, particularly in high-frequency, real-time trading scenarios.

Future research is planned to advance along three primary directions. First, traditional GARCH models will be integrated with deep learning architectures—such as LSTM and Transformer networks—to better capture non-linear volatility patterns and enhance forecasting accuracy. Second, GPU-based parallel computing will be utilized to accelerate large-scale Monte Carlo simulations, enabling real-time risk measurement at millisecond latency and supporting high-frequency trading applications. Finally, explicit incorporation of liquidity shocks and transaction costs into the model is proposed to better align derivative pricing and risk assessment with actual market mechanisms.

Authors Contribution

Chengxi Luo: conceptualization, methodology, literature retrieval & screening, investigation, data curation, model construction, software, formal analysis, validation, writing - original draft, writing - review & editing, leadership. **Yutong Liu**: conceptualization, literature retrieval & screening, writing - original draft, writing - review & editing. **Shuodong Li**: literature retrieval & screening, investigation, writing - original draft.

All the authors contributed equally.

References

[1] Roldán-Casas, J. A., García-Moreno, M. B. A procedure for testing the hypothesis of weak efficiency in financial markets: A monte carlo simulation. Journal of Financial Econometrics, 2022, 20(1): 45-67.

[2] Akintola, S. O. A Scientific Computing Analysis of Financial Black-Scholes and Monte Carlo Differential Equation: An American Option. Current Journal of Applied Science and Technology, 2024, 43(7): 181-197.

[3] Frezza, M., Bianchi, S., Pianese, A. Forecasting value-at-risk in turbulent stock markets via the local regularity of the price process. Computational Management Science, 2022, 19: 99–132.

[4] Ilić, M., Digkoglou, P. The volatility of stock market returns: Application of monte carlo simulation. Economics of Sustainable Development, 2022, 6(2): 17-30.

[5] Elman, H. C., Liang, J., Sanchez-Vizuet, T. Surrogate-based multilevel monte carlo methods for uncertainty quantification in the Grad-Shafranov free boundary problem. Computer Physics Communications, 2024, 298: 109099.

[6] Gunther, S., Nieber, P. Efficient simulation and valuation of equity-indexed annuities under a two-factor G2++ model. Journal of Risk, 2024, 16(3): 1-24.

[7] Hu, Y. Portfolio Optimization Using Machine Learning Method and Monte Carlo Simulation. Highlights in Business, Economics and Management, 2024, 41: 214–220.

[8] Zheng, Y., Xu, F. Monte Carlo simulation in barrier option pricing. Technology Entrepreneurship Journal, 2022, 36(4): 126-128.

[9] Wu, J., Jiang, Z., Zhou, W. Extreme return forecasting and trading strategy based on return interval analysis. Chinese Journal of Management Science, 2020, 28(5): 39-51.

[10] Li, Y., Sun, J. Interval prediction of China's gold futures price based on deep learning. Journal of Chongqing Technology and Business University (Natural Science Edition), 2024, 1-13.