

A Comparison of Pricing Performance of GARCH Models with Alternative Leptokurtic Innovations for American Put Option

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Abstract. The Black-Scholes option pricing model assumes constant volatility, while the volatility smile observed in financial markets indicates that volatility is time-varying. This has motivated research on volatility modeling, among which Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models are widely adopted. Leptokurtic innovation distributions are often employed to account for the excess kurtosis in asset returns. This paper compares the pricing performance of GARCH models with different innovation distributions, namely the Student's t distribution and the Generalized Error Distribution. The volatility generated from AAPL stock prices over 2014-2018 by the GARCH models, along with historical volatility, is incorporated into the binomial tree to compute American put option prices. Empirical results show that the GARCH model with Generalized Error Distribution (GED) innovations outperforms that with Student's t innovations, and both GARCH models outperform forecasts based on historical volatility. These findings provide evidence of a stronger pricing performance of GARCH models with GED innovation distribution compared to GARCH models with Student's t distribution, and highlight the importance of leptokurtic distributions on GARCH-typed option pricing.

Keywords: Option pricing, GARCH models, Leptokurtic distributions, Generalized Error Distribution (GED), Student's t distribution.

1. Introduction

The correct pricing of options is of paramount importance in financial economics. The Black-Scholes model is a well-known option pricing model in modern financial theory and is best known for providing a closed-form analytical solution for option pricing. However, one of the fundamental assumptions of the BSM model is that the underlying asset's volatility is constant over the option's lifetime. In practice, plotting implied volatility against moneyness (K/S) produces a smile-shaped curve instead of an expected flat line. Moreover, implied volatility varies across time to maturity (TTM), forming a term structure rather than remaining constant. These behaviors suggest that, in reality, volatility is not constant but depends on both moneyness and TTM.

To improve this limitation, numerous extensions have been proposed. The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) models of Engle and Bollerslev are widely used to capture time-varying volatility [1, 2]. Duan proposed a GARCH option pricing model assuming Gaussian innovations and certain investor preferences, which unified the framework of volatility under historical and pricing returns [3]. These Gaussian GARCH models have experienced some empirical success in pricing both European-style options and, after being combined with the Least Squares Monte-Carlo method, American-style options [4]. However, empirical studies show that systematic errors still occur. One of the reasons is that the market is incomplete under time-varying implied volatility, as investors can no longer replicate the contingent claims perfectly, in contrast to the Black-Scholes. Another important reason arises from empirical studies that financial asset returns usually deviate from the Gaussian assumption, as shown by Bollerslev [5]. Therefore, alternative innovation distributions are considered. Among these, leptokurtic distributions, such as the Student's t distribution and the Generalized Error Distribution (GED), are a common choice [5,6]. The underlying reason is that, in financial markets, returns are often fat-tailed. This was first documented by Mandelbrot, who pointed out that the first differences of the logarithm of cotton and common stock prices exhibit fatter tails than implied by a normal distribution [7]. Duan extended the GARCH option pricing model by allowing leptokurtic distribution (Duan used GED in that paper)

innovations, showing that leptokurtosis can significantly affect option prices [8]. A vast amount of literature has addressed to the leptokurtic distributed innovations in GARCH models, while relatively few works have focused on direct comparison between leptokurtic distributions, specifically in the context of American put options [9-11].

This paper compares the pricing performance of American put options using three different volatility inputs, and by inserting into the binomial tree model for every option, three sets of option prices are generated and are then compared with observed option prices, providing clear evidence on the relative pricing performance between Student's t GARCH and GED GARCH.

2. Methodology

2.1. Theoretical Framework

2.1.1. Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model

The GARCH model is a discrete-time model commonly used to predict volatility by analysing past price and volatility movements. Let $r_t = \log(S_t/S_{t-1})$ be the continuously compounded rates of return where S_t denotes the spot price of the underlying asset at time t . A simple assumption would be that the innovation of returns are normally distributed, write $r_t = \mu + \varepsilon_t$ where μ denotes the average return and ε_t (the innovation) is distributed as

$$\varepsilon_t = \sigma_t z_t, z_t \sim i.i.d. N(0,1) \quad (1)$$

Where z_t denotes the standardized innovation.

Bollerslev (1986) sets σ_t^2 , the conditional variance in a standard GARCH(p, q) process, to be:

$$\sigma_t^2 = \omega + \sum_{i=1}^q \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^p \beta_i \sigma_{t-i}^2, \quad (2)$$

Where

$$\begin{aligned} p &\geq 0, & q &> 0 \\ \omega &> 0, & \alpha_i &\geq 0, & i = 1, \dots, q, \\ \beta_i &\geq 0, & i &= 1, \dots, p. \end{aligned} \quad (3)$$

The σ_t^2 in GARCH (1,1) is then given by

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2 \quad (4)$$

Where

$$\omega > 0, \alpha \geq 0, \beta \geq 0. \quad (5)$$

Compared to the ARCH model, GARCH includes a new term to capture the effect of lagged conditional variances which can potentially account for the volatility clustering often observed in financial markets. The coefficient α measures the short-run impact of shocks (the news effect), while β measures the persistence of previous volatility. The set of parameter values $(\mu, \omega, \alpha, \beta)$ is estimated via the Maximum Likelihood method.

The GARCH model's proficiency in providing time-varying volatility, capturing patterns of volatility clustering, and offering flexibility in the choice of innovation distributions leads to strong forecasting performance in financial markets.

2.1.1.1 Student's t Distribution (STD)

A standard GARCH (1,1) model assumes the standardized innovations to be normally distributed. Bollerslev (1987) first introduced the STD into GARCH model trying to explain the unconditional kurtosis in the data. Here $z_t \sim t_\nu$ where $\nu > 2$ denotes the number of degrees of freedom of the STD. The probability density function of STD is given by

$$f(z_t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{(\nu-2)\pi} \Gamma(\frac{\nu}{2})} (1 + \frac{z_t^2}{\nu-2})^{-\frac{\nu+1}{2}}. \quad (6)$$

For $\nu \rightarrow \infty$ the t-distribution tends to a normal distribution whereas for any $\nu \in \mathbb{N}$ the distribution is leptokurtic. A smaller ν corresponds to fatter tails. All other equations stated in GARCH remain unchanged.

Using the STD for the innovation distribution is quite popular among GARCH volatility literature, since it is capable of capturing large kurtosis and heavy tail nature usually observed in financial time series [12-14].

2.1.1.2 Generalized Error Distribution (GED)

Generalized Error Distribution (GED) was first introduced into financial statistics by Nelson (1991) as a family of symmetric distributions with variable kurtosis, in which he proposed incorporating GED as a distribution for innovations in the GARCH model. For

$$z_t \sim GED(\nu), \quad (7)$$

The probability density function of GED is given by

$$f(z_t) = \frac{\nu}{2^{(1+\frac{1}{\nu})}\lambda\Gamma(1/\nu)} \exp [-(1/2)|z_t/\lambda|^\nu] \quad (8)$$

Where

$$\lambda = \left[2^{-\frac{2}{\nu}} \frac{\Gamma(1/\nu)}{\Gamma(3/\nu)} \right]^{\frac{1}{2}}, \quad -\infty < z < \infty, \nu > 0. \quad (9)$$

ν is a tail-thickness parameter of GED and is also called its degree of freedom. The GED reduces to the normal distribution for $\nu = 2$. For $\nu > 2$, the tail is thinner than that of the normal distribution, while thicker for $\nu < 2$. Again all other equations stated in GARCH remain unchanged.

The GED, as another leptokurtic distribution similar to the Student's t, offers greater flexibility in the thickness of the tail and has also been widely documented in the literature to fit financial returns well [15-17].

2.1.2. Historical Volatility

Historical volatility (HV) calculates the standard deviation of an underlying asset's past return over a certain period of time. In this paper, a rolling window of 30 days is chosen to measure daily HV. Again the continuously compounded rates of return is defined as $r_t = \log(S_t/S_{t-1})$, where S_t denotes the spot price of the underlying asset at time t . The daily volatility is then given by

$$\sigma_t(Daily) = \sqrt{\frac{1}{30} \sum_{i=0}^{29} (r_{t-i} - \bar{r}_t)^2} \quad (10)$$

Where $\bar{r}_t$ denotes the average return over the past 30 days. The annualized volatility, which we will use to put in the option pricing model, is obtained by multiplying the daily volatility by the square root of 252, which is approximately the number of trading days in a year:

$$\sigma_t(Annual) = \sqrt{\frac{252}{30} \sum_{i=0}^{29} (r_{t-i} - \bar{r}_t)^2} \quad (11)$$

Being a backward-looking measure, although it reflects periods of volatility patterns, HV cannot model volatility persistence explicitly compared to GARCH models, nor can it smoothly adjust in response to sudden shocks.

2.1.3. Binomial Tree

To price the options using volatility generated from alternative models, binomial tree option pricing model is used in this study. The binomial tree model is based on a discrete price model first formalized by Cox, Ross and Rubinstein [18].

First, T is divided into N equal intervals Δt . At each interval, the price will move up or down:

$$S_{t+\Delta t} = \begin{cases} S_t u, & \text{with probability } p \\ S_t d, & \text{with probability } 1 - p \end{cases} \quad (12)$$

Where u and d are defined as

$$u = e^{\sigma\sqrt{\Delta t}}, d = 1/u = e^{-\sigma\sqrt{\Delta t}} \quad (13)$$

The relation between u and d greatly reduced the number of tree nodes. Next, all the option values (put option, in this study) at maturity are calculated as $C_n = \max\{(K - S_n), 0\}$ for each stock prices S_n . The option value on each node is then obtained backwardly. At each node, the option value is given by

$$C = \max\{(K - S), (pC_u + (1 - p)C_d)e^{-r\Delta t}\}, p = \frac{e^{r\Delta t} - d}{u - d} \quad (14)$$

Where S here denotes the stock price at the current node, and C_u, C_d denote the option values at the subsequent up and down nodes respectively. The recursive process works through the tree towards the first node in time zero. The option value finally calculated at time zero is the expected option price.

2.1.4. Performance Evaluation Methods

In this study, the pricing performance of each volatility model is quantified by three statistical measures. Mean Absolute Error (MAE) captures the average size of pricing errors, while Root Mean Squared Error (RMSE) penalizes large errors more compared to MAE. The third one, bias, shows whether the models are inclined to overestimate or underestimate put prices systematically. A bias of 0 does not necessarily imply good pricing performance. A classification of data by moneyness and TTM is further made for us to measure pricing errors separately.

2.2. Data

Our sample consists of Apple Inc. (AAPL) daily closing prices and historical volatility over the period January 2014-December 2018, collected from the OptionMetrics database, accessed through Wharton Research Data Services (WRDS). American put option prices and all the other related data are collected from the same source. The paper chose Apple primarily because of its high liquidity and market activity. Option prices are computed as the midpoint of the bid-ask quotes.

Puts that have TTM less than 20 days and greater than 250 days to maturity are eliminated, as well as puts with moneyness less than 0.8 and greater than 1.2. This filter yields a total of 292,401 option observations. We classify the option data by moneyness and TTM. Denote the moneyness and TTM by $M = K/S$ and t respectively. In-the-money puts are defined as $M > 1.04$; out-of-the-money puts are $M < 0.96$; at-the-money puts have $0.96 < M < 1.04$. TTM are also divided into 3 groups: short maturity ($20 < T < 30$), medium maturity ($30 \leq T \leq 90$), and long maturity ($90 \leq T \leq 250$). Table 1 shows the amount, mean and standard deviation of put prices across 9 categories.

Table 1. Put prices description for each moneyness/ maturity group

Moneyness		Maturity					
		Less than 30		30 to 90		More than 90	
K/S		Mean	Std.	Mean	Std.	Mean	Std.
< 0.96	Put price \$	0.65	0.80	1.50	1.80	5.33	5.04
	Observations	26659		51990		29230	
0.96-1.04	Put price \$	4.99	4.87	7.50	7.07	13.98	11.71
	Observations	17732		36363		16541	
> 1.04	Put price \$	21.25	18.15	26.89	25.28	36.31	30.84
	Observations	24680		53806		35400	

For the risk-free interest rate, it uses the 3-month U.S. Treasury constant maturity rate (DGS3MO) accessed from Federal Reserve Economic Data (FRED). For each option, we use the risk-free interest rate on the valuation date for pricing. The dividend yield is obtained from Apple's investor relations official website. However, dividends are distributed discretely (quarterly), which cannot be directly put into a binomial tree. Hence, the daily dividend yield is calculated by summing up the dividend

amount over a period of one year and dividing it by the current stock price. Dividend data starting from Jan 2013 are thus incorporated in order to ensure the full coverage of the dividend yield series.

In this study, all stock prices and put prices before Apple's 7-for-1 stock split in June 2014 have been adjusted to match the stock prices and option prices after the split.

3. Results

For these three pricing models, Table 2 shows the overall pricing errors quantified by the measurements of fit discussed earlier. Table 3 categorizes the results by moneyness and TTM. The overall conclusion is that the GARCH model with GED innovation distribution outperforms the GARCH model with Student's t distribution in terms of all three measures. Furthermore, GARCH(t) tends to overestimate the put price across all categories. Also, clearly, the forecasted prices under HV have a higher MAE and RMSE than the other two GARCH-type models. However, as an exception, for short-term OTM puts, both GARCH models do not appear to outperform HV or even have a higher RMSE than HV. The underlying reason is that short-term OTM puts have limited time value plus it is low-priced, resulting in their low sensitivity on volatility changes and thereby diminishing the advantage of GARCH models. There is a clear pattern among all three models that ITM puts tend to be overpriced while OTMs tend to be underpriced. This reflects the volatility skewness, which can not be captured very well by the standard GARCH model we used. Pricing errors systematically increase with TTM, indicating the greater uncertainty caused by a longer period of time.

Table 2. Option pricing errors

	RMSE	Bias	MAE
GARCH(t)	2.81	0.16	1.19
GARCH(GED)	2.56	-0.02	1.12
HV	2.98	0.13	1.63

Table 3. Pricing errors of options across moneyness and time to maturity

Moneyness		Maturity								
		Less than 30			30 to 90			More than 90		
		RMSE	Bias	MAE	RMSE	Bias	MAE	RMSE	Bias	MAE
< 0.96	GARCH(t)	1.19	-0.05	0.49	2.19	-0.04	0.91	3.85	-0.30	1.88
	GARCH(GED)	1.08	-0.12	0.47	2.01	-0.17	0.87	3.55	-0.62	1.85
	HV	0.99	-0.08	0.54	1.80	-0.11	1.05	3.98	-0.26	2.62
0.96-1.04	GARCH(t)	2.00	0.18	1.04	3.14	0.21	1.52	4.94	0.26	2.34
	GARCH(GED)	1.85	0.04	0.97	2.88	-0.00	1.43	4.51	-0.17	2.22
	HV	2.37	0.07	1.45	3.50	0.09	2.26	5.46	0.17	3.62
> 1.04	GARCH(t)	0.88	0.07	0.48	2.13	0.26	0.91	3.75	0.81	1.65
	GARCH(GED)	0.81	0.02	0.46	1.93	0.14	0.85	3.33	0.50	1.46
	HV	1.02	0.04	0.57	1.99	0.23	1.11	4.29	0.92	2.54

4. Conclusion

It has reviewed the development of GARCH models in option pricing and examined the performance of pricing using a standard GARCH (1,1) framework by two leptokurtic innovation distributions, namely the Student's t distribution and the generalized error distribution. According to the empirical results, the GARCH-GED model provides a better fit than GARCH with Student's t innovations.

It should be noted that in this paper, the historical and pricing returns are assumed to be governed by the same model parameters. This simplification allows the direct incorporation of volatility generated in the real-world probability measure into a binomial tree under risk-neutral measure. By doing so, the added complexity of calibrating models is eliminated; however, it may come with

theoretical limitations. Further study could apply methods to transform the volatility calculated under the physical measure into the risk-neutral measure, achieving theoretical consistency.

Moreover, only the standard GARCH (1,1) is used in this study. Future research could use alternative versions of the GARCH family, such as NGARCH and EGARCH, as well as different orders, including GARCH (2,1) and GARCH (1,2), to provide a more comprehensive explanation of volatility dynamics.

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