Kelly criterion for optimizing choices in stock and portfolio markets

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Abstract. Kelly criterion is known as a useful tool in the stock market. Yet its efficiency is limited due to its simplicity because it was designed to solve binary gambling problems. Luckily, by switching its coefficients and conclude multiple outcomes in other models, the limitations can be receded or solved. This paper aims at how Kelly criterion work on stock market and portfolio statistics and how to improve it by combining with some other theories. The paper will try to use Ralph Vince's extension and combine other formulas like the Bayes model and get more improved results. By combining more adjustments, the results are shown to be more sensitive to more real factors and to be more optimized, such as integrating a vast range of data and recognizing the potential risk in the stocks' variation. In conclusion, it is promising that Kelly criterion can be more and more complete in the future with the existence of new theories that can be its adjustments.

Keywords: Kelly criterion, stock market, Ralph Vince extension, Bayes model.

1. Introduction

In 1956, John Larry Kelly Jr. described his criterion to calculate the optimal input ratio in independent and repeated bets that have a positive net income expectation. It is like betting with a super wealthy opponent who put same amount of money every time with a cheated coin in the gambler's favor [1]. Great investors like Charlie Munger and Warren Buffett had applied Kelly criterion in their portfolios and made huge successes. In those examples, Kelly criterion is proven to be useful in optimizing choices in stock markets and portfolios, the question is how does it work as stocks are far more complicated than simple independent and repeated bets.

The first application of Kelly criterion was in gambling, for most of the games are simple enough that the gamblers do not need to change the original formula much to make it work. In the scenario of a favorable bet with winning probability p>1/2, if the gambler set the betting money at the kth time as M_k, and set T_k to be the result of the trial (1 if winning and 0 if losing), then the capital C_k will conform to the equation

$$E(C_n) = C_0 + \sum_{k=1}^{n} E(M_k T_k) = C_0 + \sum_{k=1}^{n} (2p-1)E(M_k) \quad (1)$$

In this equation, the gambler can see that the more money he bet each turn, the bigger the expectation will be, however, the possibility of bankruptcy of $1-p^n$ have an expectation $\lim_{\infty} 1-p^n = 1$. So, the bankruptcy is destined when all money is betted every turn, even when the game is favorable on every turn [2]. In simple words, Kelly criterion is looking for the balanced point of maximizing expected gain and minimizing the risk of losing everything and bankrupt. For the stock market and portfolios, as the influencing factors are more various and complicated, the criterion works in a restraint way and offers a comparation between different choices rather than gives acute calculations.

This paper concludes Kelly criterion's existed uses in the stock market, and apply those on some different sets of data based on the US and CN stock market and portfolio choices. Based on these applications of the original Kelly criterion, this paper will also try to make some adjustments by combining other theories and models with Kelly criterion in order to get more optimized results. Hence, the rest of this paper is organized as following. Section 2 will present some previous works,

Section 3 will show the methods used in the optimization of stock and portfolio statistics and Kelly criterion itself, Section 4 will present the result of the optimization methods mentioned in Section 3, Section 5 will show the conclusion of the paper and make some outlooks.

2. literature review

There were existing studies for Kelly criterion's application in the stock markets, based on different limitations of the criterion and make different adjustments. Basically, because of the complicity of the real market with all the invariant winning chance, Kelly criterion cannot be applied to real cases directly in many cases. To make it work, Ralph Vince made an extension to the criterion by consider the bidding fraction under multiple sets of results instead of binary ones. Application of said extension can conclude cases of equal or nonequal probabilities of different outcomes, and offer a more generalized framework in managing money, as Kelly criterion shows as a special case in the extended theory [3].

Kelly criterion is focused on maximizing absolute wealth. By introducing relative wealth optimization, population genetics' tools and ideas can be put into consideration. In this case, when investors pursue the maximization of absolute wealth and the investment period tends to infinity, the Kelly formula is an ideal strategy to achieve the long-term optimal geometric average growth rate. However, when investors shift their focus to maximizing relative wealth (i.e., their share of the total wealth of the group), the Kelly formula is only the optimal solution under specific conditions. Research indicates that the optimal investment behavior is highly dependent on the investor's initial relative wealth (market position) and the behavior of other investors in the market. Specifically: if competitors deviate from the Kelly formula, investors in a dominant position (with more wealth) should adopt a risk deviation strategy in the same direction as their rivals; while those in a disadvantaged position (with less wealth) should adopt an opposite strategy [4].

When analyzing multivariant portfolios, the Kelly criterion is an outstanding principle of fund management, aiming to maximize the expected growth rate of investment and the median of terminal wealth. Although it is well-known in single-asset games, its application in multi-asset portfolios has received little in-depth analysis. The latest research proposes a model-free application method of the multi-Kelly formula, which only relies on the expected value (first moment) and covariance (second moment) of asset excess returns, without assuming the specific probability distribution of returns. Moreover, in response to the common constraints of no leverage and no short selling in practical investment, the study introduces a numerical algorithm based on Monte Carlo simulation. This algorithm can not only efficiently solve the optimal weights of large-scale investment portfolios but also perfectly apply to the "fractional Kelly strategies" (Fractional Kelly strategies) with lower risk and greater practicality, providing a new framework for quantitative investment that combines theoretical rigor with computational feasibility [5].

In real application of Kelly criterion in the stock market, there has been researches as well. In the article written by Rotando, L. M., & Thorp, E. O. (1992), they applied the criterion to 59 years' statistics of S &P 500 "blue chip" stocks in the US stock market. They found out that the annual excess returns of the stock to be relatively stable and it is possible to calculate various things like mean and standard deviations if assuming the AER to be independently distributed [1].

3. Methodology

This part of paper will list some methods in previous application of the criterion to real stock market statistics.

3.1. direct application

Assuming the stock can be seen as a continuous gambling game, then the outcomes can be analyzed in the following equation:

$$G(f) = \int_a^{\infty} \log(1 + fs) dF(s) \tag{2}$$

where $G(f)$ is the growth rate coefficient that Kelly criterion aims at optimizing, f is the fraction of gambler's bankroll he wagers (same every time), and F is the associated probability distribution. By putting the statistics into the equation, the gambler is able to analyze some basic characteristics of the stock, standard and mean deviations for instance. With the basic analysis, the investor can do predictions of the stock. For longer period of time, the investor can also apply machine learning to update the parameters and make the prediction more accessible and accurate [6].

3.2. Ralph Vinces' extension

Ralph Vince extends Kelly criterion with holding period return as

$$HPR_i(f) = \left(i - f\frac{-b_i}{L}\right) \tag{3}$$

As $b_i \in Z$ and $L = minb_i$. And the terminal wealth relative as the geometric mean of:

$$TWR(f) = \prod_{i=1}^{n} (HPR_i(f))^{p_i} \tag{4}$$

In the case when probabilities are unequal, by plugging in the probability f, TWR's corresponding can be generated with different f's. This is a practical way when the probability of each gamble in the stock and portfolio market is not the same. Vince's work is more applicable to real trading for it is incorporable to multiple outcomes.

3.3. Bayesian Kelly criterion

Bayesian Kelly criterion adopts the thought of random walk, and can so solve the case when the optimal policy is not considered a constant proportion.

First, user of the model needs to consider a random walk in a random environment. Set the possibility of success as θ with the density of $f_\theta(.)$, on domain (0,1), let $S_0 = 0$ and $S_n = \sum_{i=1}^{n} Z_i$, and $P(Z_i = 1|\theta) = \theta = 1 - P(Z_i = -1|\theta)$. $Y_n = (S_n + n)/2 \equiv \sum_{i=1}^{n} W_i$, $W_i(Z_i + 1)/2$. The posterior distribution of θ can be seen from a direct application of Bayes' formula and is shown as:

$$dP(\theta \leqq u|Y_n = y) = \frac{\binom{n}{y} u^y (1-u)^{n-y} f_\theta(u) du}{\int_0^1 \binom{n}{y} u^y (1-u)^{n-y} f_\theta(u) du} \tag{5}$$

It is conditioned by $(W_1, \cdots, W_n)$ depends on the sufficient statistic Y_n.

De Finetti's theorem gives another case of random walk that the set of exchangeable random variables take only the value 0 or 1.

$$dP(\theta \leqq u|S_n) = \frac{u^{\frac{S_n+n}{2}} (1-u)^{\frac{n-S_n}{2}} f_\theta(u) du}{\int_0^1 u^{\frac{S_n+n}{2}} (1-u)^{\frac{n-S_n}{2}} f_\theta(u) du} \tag{6}$$

So $\{S_n\}$is a Markov process with transition probability $P(S_{n+1} = S_n + 1|S_n) = E(\theta|S_n)$.

To reach the main objective of maximize $E(\ln V_n)$, where V_n is the money and $\{Z_n\}$ is a RWIRE, it can be shown as

$$\max_{f_1 \cdots f_n} E \ln V_n = E \sum_{i=1}^{N} \max_{f_i} E(\ln(1 + f_i Z_i|S_{i-1}) \tag{7}$$

where S_j is the value of the RW after the jth step [7].

When apply the Bayes-Kelly strategy to the Finnish stock market, it worked better than the mean-variance strategies and the $1/N$ strategy. It is confirmed that it can allocate the risky investment capital between the market index and an individual stock [8]. If the model can work under no specified

priors, no specified likelihoods and no infinite computing power then the model can fit with more circumstances [9]. Yet these premises are not easily met.

3.4. MAC method for prediction

So far, all methods this paper mentioned are purely on mathematics. The Multi-source Aggregated Classification can combine numerical features and news sentiments driven by the market into regular analysis to make it more complete [10,11]. To initialize the model, the stock price movement (y_{t+1}) at day $t+1$ is defined as 1 if the price movement is positive, otherwise, as 0. Combining the movement and the news, the model can predict the target stock's movement by analyze the news and get the news sentiment of said stock. This method also shows more sensitivity toward catastrophes like wars and pandemics [12,13]. If investors want easier ways in optimizing their portfolios, they can simply use an AI tool to process the data they want processed for they can analyze multiple datasets and simulate many scenarios [14]. Under Bayes Kelly model, for instance. With its help, investors can evaluate the long-term viability of their capital consumption and make better choices.

4. Results

This part will show the result given by applying the methods on real stock market statistics. There are existing statistic selecting and extracting techniques in the stock market such as correlation criteria, random forest, principal component analysis and autoencoder, but this paper did not apply them due to some limitations.

4.1. result in direct application

By analyzing 59 years of statistics of S&P 500 "blue chip" stocks from 1926 to 1984 with the formula, the mean μ can be calculated as 0.058 and standard deviation σ as 0.2160. The formula containing each "return" which is expressed as 1+the annual excess return ER_i is

$$\mu = \frac{1}{59}\sum_{i=1}^{59}\log(1+ER_i) = \log[\prod_{i=1}^{59}(1+ER_i)]^{\frac{1}{59}} \tag{8}$$

$$\sigma^2 = \frac{\sum_{i=1}^{59}[\log(1+ER_i)-\mu]}{58} \tag{9}$$

So, it would forecast the mean and standard deviation of the next n-year's annual excess return to be $\mu = 0.058$ and $\sigma = 0.2160\sqrt{n}$. The statistics of negative excess return $\Pr(t < \frac{0-0.058}{0.2160/\sqrt{n}})$ are shown in Table 1.

Table 1. The negative excess return of the stock based on year

Number of year n	Probability of NER	Number of year n	Probability of NER
2	0.38	20	0.13
3	0.35	25	0.10
5	0.29	30	0.08
10	0.21	35	0.06
15	0.16	40	0.05

4.2. Result of Ralph Vince's extension

Vince's method focused on finding the maximum single transaction loss, and this is the critical step of how Vince extends Kelly criterion. The maximum single transaction loss shows the maximum drawdown risk this strategy might encounter. By collecting 4 stocks' historical trading return series (TSLA, GOOG, AAPL, MSFT), the paper gets their maximum single transaction losses. Then, the paper Initialize a loop from $f = 0.01$ to $f = 1$ with the step size of 0.01, until the paper finds the f that maximizes TWR. The optimal fs of said stocks in the past 5 years is shown in Table 2.

Table 2. Optimal f of the stocks based on different methods

Stock code	Optimal f	Optimal f(with bayes)
TSLA	0.21	0.15
GOOG	0.25	0.20
AAPL	0.18	0.18
MSFT	0.20	0.22

With the f given, investors can calculate the position size they need for each stock, that would be 2.625, 3.571, 3.000, 2.85. Finally, investors can calculate the terminal wealth relative of the stocks to be 1.55, 1.72, 1.68, 1.65. With more sets of data, this model work better than the original criterion for it can integrate all those data and analyze them to get more clear sight of its future trend.

4.3. Bayesian Kelly criterion

To overcome the overfitting of the historical statistics in the optimal f method of Vince's method, this paper introduces the Bayes model to change the fixed value of win rate and expected outcome to a probability distribution like RWIRE.

In the Bayesian Kelly model, this paper sets a prior belief and constantly change it based on new observations of the market. For consecutive asset portfolio, investors can change Kelly proportion formula to $f = (\mu p - r)/\sigma^2 p$, where p means prior. Optimal fs in the Bayesian model is listed in Table 2.

And with the fs the paper can calculate the expected return rate of said stocks to be 0.195, 0.138, 0.128, 0.150. By comparing the statistics in Table 2, it can be seen that the stocks of Tesla and Google have significantly smaller optimal f with the application of Bayes model. This is because the largest loss of the said two stocks are bigger than the other two's, this factor shows bigger risk of the stock and so reduces the optimal f, and in the same time, Apple's stock didn't change shows its risk is minimal. This can also react to other risks, such as the negative relationship between the global stock market and the war between Russia and Ukraine [12] or similar relations with the COVID19 [13]. This shows this combined model have the ability to detect potential risks and inform the investors by lowering the optimal f, an ability the original criterion does not have.

4.4. MAC method

The MAC method is an individual criterion in evaluating stocks. However, it can still be used to confirm the accuracy of the outcomes of the methods this paper used. By assuming some coefficients, the Bayes Kelly model with Ralph Vince's extension needs with the stock statistics of 5 stocks in the CN stock market in the paper Multi-source Aggregated Classification for Stock Price Movement Prediction, the outcomes can be compared. By setting the winning rate as the ACC (Prediction Accuracy) in the paper mentioned, the winning rate of return $r_w = 5\%$, the failure loss rate $r_l = 4\%$, the outcome is shown in Table 3. However, this paper failed to recreate the model used in the cited paper, the statistics in Table 3 are using the model used in this paper on the cited paper's statistics to show the possibility to combine the two said models.

Table 3. The ACC, optimal f and expected return of the CN stocks

Stock code	ACC	Optimal f	Expected return
000063	70%	0.32	0.0134
000651	68.28%	0.2765	0.0118
601800	68.59%	0.2846	0.0122
000876	63.46%	0.1615	0.0067
600104	69.23%	0.3011	0.0127

From the statistics, it can be seen that the ranking of the optimal f and the expected return of the stocks follow the ranking of their ACC. This shows the Bayes Kelly model this paper used can get fairly accurate outcome that fit with other models.

5. Conclusion

Kelly criterion is introduced to be a method to optimize choices in gambles, stock markets and portfolio. However, its limitations require investors and gamblers to extend and optimize the formula before using it to get the best result. One big disadvantage of Kelly criterion is that it is aggressive with big bets that intend to be bigger when the situation seems promising. Even in the situation that the capital is theoretically infinitely divisible and the gambler can choose to bet only on cases with favorable winning rate, the Kelly gambler can still be ruined. This paradox shows that if one does not understand a formula entirely, even the most unlikely failure can happen based entirely on math itself. As J.L. Kelly wrote in his original paper: If the input symbols to a communication channel represent the outcomes of a chance event on which bets are available at odds consistent with their probabilities (i.e., fair odds), a gambler can use the knowledge given him by the received symbols to cause his money to grow exponentially. To understand the limitations of Kelly criterion and their solutions under different circumstances is the key to use it properly to maximize one's wealth. This paper had shown how can the investor combine Ralph Vince's extension, Bayes model and the MAC method to optimize Kelly Criterion and make it more sensitive to factors like potential risks and news sentiment, and also shown the possibility to combine more methods like enhanced learning and the Monte Carlo simulation to optimize the model further, in order to benefit the investors who use Kelly criterion more and more. However, this model still cannot react to important factors like how would different initial capital react differently. These kinds of problems remain to be solved by future studies.

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