Methods for Solving Indefinite and Definite Integrals Involving Trigonometric Functions

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Abstract. Integrals involving trigonometric functions represent a common yet challenging topic for students in calculus. To facilitate a deeper understanding and practical competence in solving such integrals, this paper systematically explores various solution methods, focusing primarily on seven strategic approaches. Each technique is illustrated through detailed example analyses to demonstrate its application. The study aims to help students more easily handle integrals containing trigonometric functions, thereby increasing their learning interest and strengthening comprehensive mathematical skills. Success in this area requires keen observation to identify the features of different integral forms and to select suitable methods based on their specific characteristics. Furthermore, solving these integrals reinforces fundamental concepts of calculus, as they often involve combining integration techniques—such as substitution and integration by parts—with trigonometric identities. Additionally, this process enhances problem-solving abilities by teaching how to simplify complex expressions using identities (e.g., Pythagorean identities, double-angle formulas) and how to choose effective integration strategies. This work will benefit readers interested in calculus, as it aims to deepen their understanding of integral techniques.

Keywords: trigonometric function, definite integral, indefinite integral, integration by part.

1. Introduction

Integration is an important part of advanced mathematics. In the 7th century, the scientific revolution (such as the need for "instantaneous rate of change" and "area under the curve" in astronomy and mechanics) led to the transformation of integration from an "intuitive method" to a systematic theory. The core of this was the "fundamental theorem of calculus" proposed by Newton and Leibniz: In 1684, Leibniz used the symbol "∫" (derived from the first letter of the Latin word "summa" (total sum)) for the first time in his work to clearly define integration as "the sum of infinitesimal quantities", and independently discovered the fundamental theorem of calculus [1]. This marked the birth of calculus. In the 19th century, mathematicians, in an effort to rectify the logical errors in calculus, replaced "infinitesimal quantities" with "limits" and established a rigorous integral theory. This directly gave rise to real analysis. As mathematics became more abstract, integration no longer confined to "functions on the real number field", but expanded to the complex number field and function spaces, thereby generating complex analysis and functional analysis respectively, and profoundly influencing the solution of differential equations [2,3].

Integrals involving trigonometric functions are particularly common and fundamental in the field of integration problems. Please. Due to its complex and diverse calculation methods, it is also one of the difficult points that students find hard to master. In order to make it easier for students to master

the integration involving trigonometric functions, this paper discusses this type of integration. Mainly, several calculation methods are proposed, and each method is elaborated and explained through example analysis [4]. This is beneficial to enhancing students' interest in learning advanced mathematics and cultivating their logical thinking ability. In short, the history of integrals has undergone a transformation from "concrete calculation" to "abstract theory". Each breakthrough has given rise to new branches of mathematics and has become a core tool in fields such as physics, engineering, and computer science [5,6].

The comprehensive problems involving trigonometric functions play a significant role in numerous practical scientific research fields and lay an important theoretical foundation for many scientific research areas. For instance, in the process of studying physical problems, Gaussian integration was employed; when dealing with optical problems, Fourier integrals and Bessel integrals were utilized.

This paper aims to show the ways which solves the integral problems with trigonometric function. Section 2 will show five different ways which could be used to deal with the problems with trigonometric functions. Section 3 will explain some difficult examples in the field of definite integrals that involve trigonometric functions.

2. Calculation of Indefinite Integral

2.1. Trigonometric Identities

This method is mainly applicable to integrals where the integral contains trigonometric functions. The typical applicable formula is $\sin(\xi+\eta)=\sin\xi\cos\eta+\cos\xi\sin\eta$ [7]. Besides, other formulas include $sin^2\zeta+cos^2\zeta=1$, $\cos^2\zeta=\frac{1+cos2\zeta}{2}$, $sin^2\zeta=\frac{1-cos2\zeta}{2}$, and $tan^2\zeta+1=sec^2\zeta$. The indefinite integral relations are

$$\int \cos x\,dx=\sin x+c, \int \sin x\,\mathrm{d}x=-\cos x+c \tag{1}$$

Example 1. Calculate $\int \cos^2 x dx$.

Solution: This integral is calculated as follows:

$$\int \cos^2 x dx=\int \frac{}{2} dx=\frac{}{2}\left(\int dx+\int \cos 2x dx\right) \\ =\frac{1}{}\left[\int dx+\int \cos 2x\, d(2x)\right]=\frac{1}{}x+\frac{1}{}\sin 2x+c \tag{2}$$

Example 2. Calculate $\int x^2\cos x\,dx$

Solution: This integral is calculated as follows:

$$=x^2\sin x-2\int x\sin x\,dx=x^2\sin x-2\int x d(-\cos x) \tag{3}$$

2.2. Integration by Parts

The formula for the integration by parts is $\int udv=uv-\int vdu$. The key to using the integration by parts formula lies in the selection of u and v. To facilitate the application of the integration by parts method, regarding the selection of u, the following three cases are mainly considered. The case 1 is like $\int x^a\sin\beta x dx$, $\int x^a\cos\beta x dx$, $\int x^a e^{\beta x}dx$, in which one can let $x^a=u$. The case 2 is like $\int x^a arc\sin x\,dx$, $\int x^a arc\tan x$, $\int x^a ln\beta x dx$, in which one could let $\arcsin x$, $\arctan x$, $\ln\beta x=u$. The case 3 is like $\int e^{ax}\sin\beta x dx$, $\int e^{ax}\cos\beta x\,dx$, in which one can let $e^{ax}=u$ or $\sin\beta x$, $\cos\beta x=u$ [8]. However, it is necessary to pay attention to the consistency of u.

Example 3. Calculate $\int e^x \cos x \, dx$.

Solution: Obviously, the integration is the product of an exponential function and a cosine function. According to Case 3, both e^x and $\cos x$ can be set as u. Let people assume $\cos x$ as u. Firstly,

$$I = \int e^x \cos x \, dx = \int \cos x \, de^x \tag{4}$$

Using integration by parts and letting $\cos x$ is u,

$$I = e^x \cos x - \int e^x d \cos x = e^x \cos x - \int e^x \sin x \, dx \tag{5}$$

Next, one can apply integration by parts again. The selection of u must be consistent with the first time. Let sinx be u, then

$$I = e^x \cos x + \int \sin x \, de^x = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx \tag{6}$$

After moving terms, one finally get that

$$I = \int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c \tag{7}$$

2.3. Substitution Method with Radicals

This method is mainly applicable to integrals where the integration contains square roots. By using the radical substitution method, the square root can be removed from the integral expression, simplifying the integration.

Example 4. Calculate $\int \sqrt{2} \cos \sqrt{x} \, dx$

Solution: Let $\sqrt{x} = u$ then $\mathrm{x} = \mathrm{u}^2$ and $\mathrm{dx} = 2\mathrm{udu}$. Thus,

$$I = \int \sqrt{x} \cos \sqrt{x} \, dx = \int u \cos u \times 2u du = 2 \int u^2 \cos u \, du \tag{8}$$

From Example 2, it is found that

$$\begin{aligned} I &= \int \sqrt{x} \cos \sqrt{x} \, dx = 2u^2 \sin u + 4u \cos u - 4 \sin u + c \\ &= 2\mathrm{x} \sin \sqrt{\mathrm{x}} + 4\sqrt{\mathrm{x}} \cos \sqrt{x} - 4 \sin \sqrt{\mathrm{x}} + c \end{aligned} \tag{9}$$

2.4. Weierstrass Substitution

In this example, let $\tan \frac{\mathrm{x}}{2} = \mathrm{t}$, one can transform the integration to make it suitable for integration.

Example 5. Calculate $\mathrm{I} = \int \frac{1}{4+5\cos x} dx$

This integral can be solved by substitution. Let $\tan \frac{x}{2} = t$, so $\cos x = \frac{1-\mathrm{t}^2}{1+\mathrm{t}^2}$ and $\mathrm{dx} = \frac{2}{1+\mathrm{t}^2} \mathrm{dt}$. Thus, the integral can be rewritten as

$$I = \int \frac{1}{4+5\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = \int \frac{2}{2+4t^2+5-5t^2} dt = 2 \int \frac{1}{9-\mathrm{t}^2} \mathrm{dt} \tag{10}$$

The integration can be separated as follows: $\frac{1}{9-\mathrm{t}^2} = \frac{1}{(3-\mathrm{t})(3+\mathrm{t})} = \frac{1}{3-\mathrm{t}} - \frac{1}{3+\mathrm{t}}$. Consequently,

$$I = \frac{1}{3} \ln \left| \frac{3+t}{3-t} \right| = \frac{1}{3} \ln \left| \frac{3+\tan \frac{x}{2}}{3-\tan \frac{x}{2}} \right| \tag{11}$$

2.5. Utilizing Special Factors

If the integration contains special factors, one can try to transform the integrand using the following formula. The details are as follows: (1) If the integration contains a special factor x, it can be transformed using the formula $\mathrm{xdx} = \frac{1}{2} \mathrm{dx}^2$. (2) If the function contains the factor x^2, one can use the formula $x^2 dx = \frac{1}{3} dx^3$ to transform the integration.

Example 6. Calculate $\int \frac{sec^2 x}{\sqrt{2tanx+1}}$

Solution: The integral can be calculated as

$$I = \int \frac{}{\sqrt{2\tan x + 1}} = \int \frac{}{\sqrt{2\tan x + 1}} dtanx$$
$$= \frac{1}{} \int \frac{1}{} d(2\tan x + 1) = \sqrt{2\tan x + 1} + c \tag{12}$$

3. Calculation of Definite Integral

3.1. Calculate $\int_0^{\frac{\pi}{3}} \frac{\sin x}{1-\sin x} dx$

Since the question is not very complicated, one can consider directly applying the properties of trigonometric functions [9]. Thus, it is found that

$$I = \int_0^{\frac{\pi}{3}} \frac{\sin x(1+\sin x)}{(1-\sin x)(1+\sin x)} \mathrm{dx} = \int_0^{\frac{\pi}{3}} \frac{\sin x+\sin^2 \mathrm{x}}{1-\sin^2 \mathrm{x}} \mathrm{dx} = \int_0^{\frac{\pi}{3}} \frac{\sin x+\sin^2 \mathrm{x}}{\cos^2 \mathrm{x}} \mathrm{dx} \tag{13}$$

After splitting the numerator, it is calculated as

$$I = -\int_0^{\frac{\pi}{3}} \frac{\mathrm{d}\cos \mathrm{x}}{\cos^2 \mathrm{x}} + \int_0^{\frac{\pi}{3}} \tan^2 \mathrm{x}\, \mathrm{dx} = \frac{1}{\cos\frac{\pi}{3}} - \frac{1}{\cos 0} + \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) = 1 + \sqrt{3} - \frac{\pi}{3} \tag{14}$$

3.2. Calculate $\int_0^{\pi} \frac{x}{(1+cos^2 x)^3} dx$

The author shall start from considering the general case

$$I_n = \int_0^{\pi} \frac{x}{(1+cos^2 x)^n} = \int_{\frac{\pi}{2}}^{\pi} \frac{\mathrm{x}}{(1+\cos^2 \mathrm{x})^{\mathrm{n}}} + \int_0^{\frac{\pi}{2}} \frac{x}{(1+cos^2 x)^n} = \pi \int_0^{\frac{\pi}{2}} \frac{x}{(1+cos^2 x)^n} dx \tag{15}$$

Using $\tan \mathrm{x}$ to replace $\cos \mathrm{x}$, then

$$I_n = \pi \int_0^{\frac{\pi}{2}} \frac{1}{(2+\tan^2 \mathrm{x})^{\mathrm{n}}} \frac{1}{\cos^{2\mathrm{n}} \mathrm{x}} dx = \pi \int_0^{\frac{\pi}{2}} \frac{(1+\tan^2 \mathrm{x})^{\mathrm{n}-1}}{(2+\tan^2 \mathrm{x})^{\mathrm{n}}} \frac{1}{\cos^2 \mathrm{x}} dx \tag{16}$$

Let $u = \tan x, x \in (0, \frac{\pi}{2})$, then

$$I_n = \pi \int_0^{+\infty} \frac{(1+\mathrm{u}^2)^{\mathrm{n}-1}}{(2+\mathrm{u}^2)^{\mathrm{n}}} \mathrm{du} \tag{17}$$

This integral can be solved by substitution [10]. Let $\mathrm{u} \to \sqrt{2}\mathrm{u}$, then

$$I_n = \frac{\sqrt{2}\pi}{2^{\mathrm{n}}} \int_0^{+\infty} \frac{(1+2\mathrm{u}^2)^{\mathrm{n}-1}}{(1+\mathrm{u}^2)^{\mathrm{n}}} \mathrm{du} \tag{18}$$

One can use substitution again. Let $\mathrm{u} = \tan\theta, \theta \in (0, \frac{\pi}{2})$, it is found that

$$I_n = \frac{}{2^{\mathrm{n}}} \int_0 \frac{}{(1+\tan^2\theta)^2} \frac{}{\cos^2\theta} d\theta = \frac{}{2^{\mathrm{n}}} \int_0 (1+\sin^2\theta)^{\cdot} d\theta$$
$$- \frac{\sqrt{2}\pi}{} \sum^{\mathrm{n}=1} \binom{\mathrm{n}-1}{} \frac{(2\mathrm{k}-1)!!}{} \frac{\pi}{} \tag{19}$$

Thus, when $n = 3$, one finds that

$$\mathrm{I}_3 = \frac{\sqrt{2}\pi^2}{2^4} \sum_{\mathrm{k}=0}^{2} \binom{2}{\mathrm{k}} \frac{(2\mathrm{k}-1)!!}{(2\mathrm{k})!!} = \frac{\sqrt{2}\pi^2}{2^4} \left(1 \times 1 + 2 \times \frac{1}{2} + 1 \times \frac{3}{8}\right) = \frac{19\sqrt{2}\pi}{128} \tag{20}$$

3.3. Applications in Different Areas

In engineering, addressing "quantitative design in real-world scenarios" involves translating abstract physical principles into implementable parameters—such as component strength and fluid flow rate. Definite integrals serve as a "bridge from theory to application." For instance, in mechanical engineering, stress and strength analyses of components like shafts and beams require the calculation of the moment of inertia—a key parameter quantifying resistance to bending—via definite integrals. Consider a rectangular cross-section beam: its moment of inertia about the neutral axis is given by $I_z = \int y^2 dA$, where A denotes the cross-sectional area and y represents the distance from the differential area dA to the neutral axis. A higher moment of inertia corresponds to greater bending resistance. Engineers leverage such integrals to optimize cross-sectional dimensions; for example, I-shaped beams exhibit higher inertia than rectangular sections, leading to more efficient material use.

In economics, definite integration plays a central role in analyzing the relationship between marginal and total quantities. The marginal quantity (e.g., marginal cost or revenue) is the derivative of the corresponding total quantity (e.g., total cost or revenue). Definite integration enables the reverse process—recovering total quantities from marginal functions—thereby supporting decision-making and optimization. In statistics, definite integrals are fundamental for constructing quantitative models of probability distributions and expectations. They are essential in characterizing continuous random variables such as height, weight, or time and are used to compute core metrics including probabilities and expected values.

4. Conclusion

This article mainly adopts the form of specific analysis through examples to present various methods for calculating integrals containing trigonometric functions. However, each calculation method is not universal and has its own characteristics of application. Therefore, in the process of using these methods, one needs to select the appropriate method based on the characteristics of the integration, aiming at a specific goal. This not only helps to enhance students' interest in learning advanced mathematics, encourage them to actively participate in the teaching process, but also benefits the cultivation of students' logical thinking ability and comprehensive ability. Trigonometric functions are merely the starting point and foundation in the field of integration. Starting with trigonometric functions, one can gradually delve deeper into the mysteries of integration and study the domains of exponential and logarithmic functions, and even reach more challenging special functions such as the Gaussian function and the gamma function. Whether it is economics that is beneficial to national management, or engineering disciplines such as physical chemistry that are related to national construction, whether it is research or experimentation, mathematical models and calculations are indispensable. Therefore, mathematics has become an indispensable theoretical foundation and tool for various industries.

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