Mathematical Modeling and Algorithm Application
ISSN: 3006-0842 | Vol. 2, No. 2, 2024

Adomian decomposition method for solving fuzzy fractional Volterra-Fredholmintegro-differential equations

Qin Chen

School of Mathematics and Information, China West Normal University, Nanchong, 637009, China
* **Corresponding author**: Qin Chen (Email: 1751743987@qq.com)

Abstract: This paper mainly studies the fuzzy nonlinear fractional Volterra-Fredholm integro-differential equations based on fuzzy Caputo derivative under the generalized Hukuhara difference. By usingSchauder fixed point theorem, the existence of solutions are proved. Because of the good convergence and convenient calculation of the Adomian decomposition method (ADM), we expand the nonlinear part of the equation into the Adomian polynomial of infinite series, and then construct the iterative sequence of the numerical solution of the equation. The effectiveness and applicability of ADM are verified by numerical examples.

Keywords: Fuzzy fractional Volterra-Fredholm integral equations; fixed point theorem; Adomian decomposition method.

1. Introduction

With the rapid development of fractional integral differential, many phenomena, including fluid flow, electrical network, fractal theory, control theory, optics, biology, chemistry, etc., can be transformed into fractional integral differential equations to be effectively solved. The application of fractional derivative was first proposed by Abel to solve the integral equation in Tautochrone problem [1]. The concept of Riemann-Liouville fractional differential equations was first introduced by Agarwal et al. [2]. Rahaman et al. used several different models in their research, such as the generalized fractional economic production quantity model [3]. However, in the mathematical modeling of some practical problems, the parameters are often uncertain. In order to overcome the influence of parameter uncertainty, the concept of fuzzy number is introduced. Zadeh introduced the concept of fuzzy numbers and arithmetic operations in the article [4,5], and then Mizumoto and Tanaka [6] further enriched it. Dubois and Prade [7] proposed the concept of fuzzy function set. The Hukuhara derivative and fuzzy initial value problems of fuzzy valued functions are proposed in [8] and studied in [9]. Stefanini and Bede [10] proposed an extension of the Hukuhara difference. In the past decade, many scholars have also explored the fuzzy fractional integral and differential, and introduced some new concepts about fuzzy fractional integral, differential and integral differential, which are used to solve the integral and differential problems in physics, medical electrochemistry, economics, electromagnetism, control theory and viscoelasticity [11–16].

In 2011, Lupulescu and Arshad [17,18] studied the existence and uniqueness of solutions to fuzzy initial fractional differential equations with the following form

$$D^{\alpha}u(t)=f(t,u(t)),\ \lim_{t\to 0_+}t^{1-\alpha}u(t)=v_0 \tag{1.1}$$

where D^{α} is the α order fractional derivative defined in the sense of Riemann-Liouville and $0<\alpha<1$, $f\in C([0,a]\times E,E)$ is a continuous function, $v_0\in E$ is a fuzzy number, E represents all fuzzy number spaces, and the meaning of E in the following content is the same as here.

In 2020, Ahmad and Ullah [20] studied the existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equations

$$
\begin{aligned}
{}^{c}D^{\alpha}\tilde{v}(y,r)&=g(y)+a(y)\tilde{v}(y,r)+\int_0^y M_1(y,\xi)N_1(\tilde{v}(\xi,r))d\xi\\
&+\int_0^1 M_2(y,\xi)N_2(\tilde{v}(\xi,r))d\xi,
\end{aligned}\tag{1.2}
$$

with initial condition $\tilde{v}(0,r)=\left[\underline{v}(0,r),\overline{v}(0,r)\right]$, for $r\in[0,1]$, ${}^{c}D^{\alpha}$ is the α order Caputo fractional derivative, and $\tilde{v}(y,r)$ is a fuzzy function. $g:U\to R$ and $M_i:U\times U\to R(i=1,2)$ are continuous functions and $U=[0,1]$, $N_i:R\to R(i=1,2)$ are Lipschitz continuous functions.

Based on the above research, we will study the following fuzzy nonlinear frac- tional Volterra-Fredholm integro-differential equations

$$
\begin{aligned}
{}^{c}D^{\gamma}w(x,r)&=g(x,r)+a(x)w(x,r)\\
&+\int_0^x k_1(x,t)F_1(w(t,r))dt+\int_0^1 k_2(x,t)F_2(w(t,r))dt,
\end{aligned}\tag{1.3}
$$

$$w(0,r)=[\underline{w}(0,r),\overline{w}(0,r)],\tag{1.4}$$

${}^{c}D^{\gamma}$ denotes the Caputo fractional generalized derivative of order γ, $0<\gamma\le 1$, $w(x,r)=[\underline{w}(x,r),\overline{w}(x,r)]$ is a fuzzy function. $k_1\in(\Delta,E),\Delta=\{(x,t):0\le t\le x\le 1\}$, $k_2\in C([0,1]\times[0,1],E),F_i\in C([0,1]\times E,E)(i=1,2)$ and E denotes the fuzzy number space.

Because the analytical solutions of some integro-differential equations are difficult to obtain, in addition to the study of the existence and uniqueness of the solutions of integro-differential equations, scholars also use different numerical methods to solve integro-differential equations. Hamoud and Ghadle [21] studied the improved Laplace decomposition method for solving fractional Volterra-Fredholm integro-differential equations. Furthermore, they used the modified ADM to solve the fractional Volterra-Fredholm integro-differential equations in [22] and used the

Homotopy analysis method (HAM) to solve fuzzy Volterra-Fredholm integral-differential equations in [23]. Bani Issa et al. [24] also studied the second kind of fuzzy integro-differential equations by using ADM. Because of the convergence speed of ADM and the small amount of calculation, this paper applies this method to numerically solve Eqs.(1.3)-(1.4).

Based on previous studies, this paper has two main contributions. ADM is often used to solve linear and nonlinear differential and integral problems, and the method has fast convergence speed and small calculation amount. Therefore, this paper applies ADM to the numerical solution of nonlinear fuzzy fractional Volterra-Fredholm integro-differential equations. Another contribution is to prove the existence and uniqueness of the solution of the equation by using Banach fixed point theorem and Schauder fixed point theorem.

This work is organized as follows: Section 2 It is mainly about the preliminary knowledge of fuzzy definition and theorem. Section 3 the iterative scheme of ADM and its solution is introduced. In Section 4, the main results of existence, uniqueness of solutions for nonlinear fuzzy fractional Voltera-Fredholm integro-differential equations are given. Several numerical examples are given in Section 5. Finally, Section 6 gives a brief conclusion.

2. Preliminaries

In this section, we will give the related concepts of fuzzy numbers, and some important theorems and symbols used in the article[25-35].

Definition2.1. [28,29] $\mathcal{F}(\mathrm{R})$ denotes the set of all fuzzy sets on R. Let $h \in \mathcal{F}(\mathrm{R})$, if h satisfies

(i) h is a normal fuzzy set, i.e., there exists $s_0 \in \mathrm{R}$ such that $h(s_0) = 1$,

(ii) h is a convex fuzzy set, i.e., $h(\delta s_1 + (1-\delta)s_2) \geq \min\{h(s_1), h(s_2)\}$ for all $s_1, s_2 \in \mathrm{R}$ and $\delta \in [0,1]$,

(iii) h is an upper semi-continuous function,

(iv) The closure of the support of h is compact, i.e., $[h]^0$ is compact,

then h is called as a fuzzy number. The set of all fuzzy numbers is known as the fuzzy number space, denoted by E.

Definition2.2. [34] Given $0 \leq r \leq 1$, a fuzzy number h in parametric form is represented by an ordered function pairs $\left(\underline{h}(r), \overline{h}(r)\right)$ satisfying

(i) $\underline{h}(r)$ is a bounded left continuous non decreasing function,

(ii) $\overline{h}(r)$ is a bounded left continuous non increasing function,

(iii) $\underline{h}(r) \leq \overline{h}(r)$.

For $h = (\underline{h}, \overline{h}), v = (\underline{v}, \overline{v}) \in E$ and $\delta \in \mathrm{R}$, the sum of $v + h$ and the scalar multiplication δh can be defined by

$$\left(\underline{v+h}\right)(r) = \underline{v}(r) + \underline{h}(r), \quad \left(\overline{v+h}\right)(r) = \overline{v}(r) + \overline{h}(r), \quad \forall r \in [0,1],$$

and

$$\delta h = \begin{cases} \left(\delta\underline{h}, \delta\overline{h}\right), \delta \geq 0, \\ \left(\delta\overline{h}, \delta\underline{h}\right), \delta \leq 0. \end{cases}$$

Definition2.3. [29]For any two fuzzy numbers w and h, defin $D_r: E \times E \to R^+ \cup \{0\}$ by

$$D_r(v,h) = \sup_{r \in [0,1]} \max\{|\underline{v}(r) - \underline{h}(r)|, |\overline{v}(r) - \overline{h}(r)|\}.$$

where $v = [\underline{v}(r), \overline{v}(r)], h = [\underline{h}(r), \overline{h}(r)]$. It has the following useful propertie.

For $\forall w, h, v, \alpha \in E$, there are

(i) (E, D_r) is a complete metric space,

(ii) $D_r(w+v, h+v) = D_r(w,h)$,

(iii) $D_r(w,h) \leq D_r(w,v) + D(v,h)$,

(iv) $D_r(\alpha w, \alpha h) = \| \alpha \| D_r(w,h)$, (see [32,33]).

(v) $\| w \| = D_r(w, \tilde{0})$, (see [30]).

(vi) $D_r\left(\int_J w(s)ds, \int_J h(s)ds\right) \leq \int_J D_r\left(w(s), h(s)\right)ds$,

(vii) $D_r(w \tilde{*} h, \tilde{0}) = D_r(w, \tilde{0}) D_r(h, \tilde{0})$ with the fuzzy multiplication $\tilde{*}$ is based on the extension principle that can be proved by α -cuts of fuzzy numbers $w, h \in E$. Here $\tilde{0} \in E$ is defined by (see [35])

$$\tilde{0}(t) = \begin{cases} 1, s = 0, \\ 0, elsewhere. \end{cases}$$

Lemma 2.1. [36] The initial value problem of fuzzy fractional integro-differential equation (1.4) is equivalent to one of the following integral equations

Case I If $w(x)$ be cf [(i)-gH]-differentiable, then

$$w(x,r) = w(0,r) + \frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}[g(s,r) + a(s)w(s,r) + \int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau]ds.$$

Case II If $w(x)$ be cf [(ii)-gH]-differentiable, then

$$w(x,r) = w(0,r) \Theta \frac{-1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}[g(s,r) + a(s)w(s,r) + \int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau]ds.$$

Case III If there exists a switching point t0 ∈ [0, 1] such that $w(x)$ is cf [(i)-gH]- differentiable on [0, t0] and cf [(ii)-gH]-differentiable on (t0, 1), then

$$\begin{cases} w(x,r) = w(0,r) + \dfrac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}[g(s,r) + a(s)w(s,r) \\ + \int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau]ds, x \in [0,t_0], \\ w(x,r) = w(0,r) \Theta \dfrac{-1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}[g(s,r) + a(s)w(s,r) \\ + \int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau + \int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau]ds, x \in [t_0,1]. \end{cases}$$

(

3. Adomian decomposition method

Considering Eq.(1.3) and initial condition (1.4), the operator is applied to both sides of Eq.(1.3). It can be obtained from Lemma 2.1 that

$$w(x,r) = w(0,r) + J^{\gamma}[g(x,r) + a(x)w(x,r) + \int_0^x k_1(x,t)F_1(w(t,r))dt + \int_0^1 k_2(x,t)F_2(w(t,r))dt]. \tag{3.1}$$

The solution $w(x,r)$ of Eq.(1.3) can be expressed by the infinite series

$$w(x,r) = \sum_{i=0}^{\infty} w_i(x,r). \tag{3.2}$$

The ADM identifies the nonlinear functions F1 and F2 by the decomposing series

$$F_1(x,w(x,r))=\sum_{n=0}^{\infty}A_n(x,r),$$
$$F_2(x,w(x,r))=\sum_{n=0}^{\infty}B_n(x,r), \tag{3.3}$$

where the Adomian polynomials An and Bn are given by [37–41]

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F_1(\sum_{i=0}^{n}\lambda^i w_i)\right]_{\lambda=0},$$
$$B_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F_2(\sum_{i=0}^{n}\lambda^i w_i)\right]_{\lambda=0},\ n=0,1,2,\cdots.$$

Substituting Eq.(3.2)-(3.3)into Eq.(3.1), the iterative formula is

$$w_0(x,r)=w(0,r)+J^{\gamma}[g(x,r)],$$
$$w_{i+1}(x,r)=J^{\gamma}[a(x)w_i(x,r)+\int_0^x k_1(x,t)A_i(w(t,r))dt$$
$$+\int_0^1 k_2(x,t)B_i(w(t,r))dt],\ i=0,1,2\cdots. \tag{3.4}$$

4. Existence and uniqueness of the solution

Fixed point theorem will be used to prove the existence and uniqueness of the solution of Eq.(1.4). The theoretical proof will depend on the following hypotheses.

(G_1) $g(x,r),a(x):[0,1]\to E$, assume that g(x), f(x) are continuous functions, and write $\zeta=\sup_{x\in[0,1]}|a(x)|$.

(G_2) There exist two continuous functions $P_1>0,P_2>0$ satisfying

$$P_1=\sup_{x\in[0,1]}\int_0^x|k_1(x,t)|dt<\infty,P_2=\sup_{x\in[0,1]}\int_0^1|k_1(x,t)|dt<\infty.$$

we will use Schauder fixed point theorem to prove the existence of solutions of Eq.(1.4)

Theorem 4.1. Suppose that the hypotheses (G_1), (G_2) hold, if

$$\frac{\zeta}{\Gamma(\gamma+1)}<1, \tag{4.1}$$

then Eqs.(1.3)-(1.4) *has at least one solution* $w(x,r)\in C_F[0,1]$.

Proof. Let the definition of operator $T:C_F[0,1]\to C_F[0,1]$,

$$(Tw)(x,r)$$
$$=w(0,r)+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}g(s,r)ds \tag{4.2}$$
$$+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}a(s)w(s,r)ds$$
$$+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$
$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds.$$

(1) We first prove that T is a completely continuous operator. It can be done in the following three steps.

(i) The first step is to prove that T is a continuous operator. Let $w_n(x,r)$ be a sequence satisfying $w_n(x,r)\to w(x,r)$ on, for any $w_n(x,r),w\in E,x\in[0,1]$.

Using the properties of the metric D, and the assumptions, there is

$$D(Tw_n(x,r),Tw(x,r)) \tag{4.3}$$
$$=D(w(0,r)+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}g(s,r)ds$$
$$+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}a(s)w_n(s,r)ds$$
$$+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w_n(\tau,r))d\tau+\int_0^1 k_2(s,\tau)F_2(w_n(\tau,r))d\tau)ds,$$
$$w(0,r)+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}g(s,r)ds+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}a(s)w(s,r)ds$$
$$+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds)$$
$$\le D(\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}a(s)w_n(s,r)ds,\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}a(s)w(s,r)ds)$$
$$+D(\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w_n(\tau,r))d\tau)ds,$$
$$\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau)ds)$$
$$+D(\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}(\int_0^1 k_2(s,\tau)F_2(w_n(\tau,r))d\tau)ds,$$
$$\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}(\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds)$$
$$\le\frac{1}{\Gamma(\gamma)}\int_0^x|(x-s)^{\gamma-1}|\sup_{s\in[0,1]}|a(s)|D(w_n(x,r),w(x,r))ds$$
$$+\frac{1}{\Gamma(\gamma)}\int_0^x|(x-s)^{\gamma-1}||\int_0^s|k_1(s,\tau)|d\tau D(F_1(w_n(\tau,r)),F_1(w(\tau,r)))ds$$
$$+\frac{1}{\Gamma(\gamma)}\int_0^x|(x-s)^{\gamma-1}||\int_0^1|k_2(s,\tau)|d\tau D(F_2(w_n(\tau,r)),F_2(w(\tau,r)))ds$$
$$\le\frac{\zeta}{\Gamma(\gamma+1)}D(w_n(x,r),w(x,r))+\frac{P_1}{\Gamma(\gamma+1)}\sup_{\tau\in[0,1]}D(F_1(w_n(\tau,r)),F_1(w(\tau,r)))$$
$$+\frac{P_2}{\Gamma(\gamma+1)}D(F_2(w_n(\tau,r)),F_2(w(\tau,r))),$$

Because $\lim_{n\to\infty}w_n(x,r)=w(x,r)$ and F1 and F2 are continuous functions, we have

$$D(Tw_n(x,r),Tw(x,r))\to 0, \tag{4.4}$$

It is deduced from Eq.(4.4) that T is a continuous operator.

(ii) The second step is to prove that the operator T maps a bounded set into a bounded set on C_F [0; 1].

Indeed, $\forall\lambda>0$, we need to prove that if there exists a constant $\alpha>0$, such that for each $w(x,r)\in B_\lambda=\{w(x,r)\in E:D(w(x,r),\tilde{0})\le\lambda\}$,we can find a such that $D(Tw(x,r),\tilde{0})\le\alpha$.

$$D(Tw(x,r),\tilde{0})$$
$$=D(w(0,r)+\frac{1}{\Gamma(\gamma)}\int_0^x(x-s)^{\gamma-1}g(s,r)ds \tag{4.5}$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}a(s)w(s,r)ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x (x-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds,\tilde{0})$$

$$\le D(w(0,r),\tilde{0})+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(g(s,r),\tilde{0})ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(a(s)w(s,r),\tilde{0})ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau,\tilde{0})ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^1 \left|(x-s)^{\gamma-1}\right| D(\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau,\tilde{0})ds$$

$$\le D(w(0,r),\tilde{0})+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| D(g(s,r),\tilde{0})ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| \sup_{s\in[0,1]}\left|a(s)\right| D(w(s,r),\tilde{0})ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^x \left|(x-s)^{\gamma-1}\right| \int_0^s \left|k_1(s,\tau)\right| d\tau \sup_{\tau\in[0,1]} D(F_1(w(\tau,r)),\tilde{0})ds$$

$$+\frac{1}{\Gamma(\gamma)}\int_0^1 \left|(x-s)^{\gamma-1}\right| \int_0^1 \left|k_2(s,\tau)\right| d\tau \sup_{\tau\in[0,1]} D(F_2(w(\tau,r)),\tilde{0})ds.$$

Let $\mu_1 := \sup_{0\le\tau\le1} D(F_1(w(\tau,r)),\tilde{0})+1$, $\mu_2 := \sup_{0\le\tau\le1} D(F_2(w(\tau,r)),\tilde{0})+1$, $\sigma = \sup_{s\in[0,1]} D(g(s,r),\tilde{0})$ Combining with the assumption give

$$D(Tw(x,r),\tilde{0})$$

$$\le \frac{\sigma}{\Gamma(\gamma+1)}+\frac{\zeta\lambda}{\Gamma(\gamma+1)}+\frac{\mu_1 P_1}{\Gamma(\gamma+1)}+\frac{\mu_2 P_2}{\Gamma(\gamma+1)}+D(w(0,r),\tilde{0}) \quad (4.6)$$

$$:= \alpha.$$

Therefore $\forall w(x,r)\in B_\lambda$,, which implies that $TB_\lambda \subset B_\alpha$.

(iii) The third step is to prove that T maps a bounded set to an equicontinuous set. Let B_λ be defined as in the step (ii). For $\forall w(x,r)\in B_\lambda, x_1\le x_2, x_1,x_2\in[0,1]$

$$D(Tw(x_2,r),Tw(x_1,r))$$

$$= D(w(0,r)+\frac{1}{\Gamma(\gamma)}\int_0^{x_2}(x_2-s)^{\gamma-1}g(s,r)ds$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}a(s)w(s,r)ds \quad (4.7)$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds,$$

$$w(0,r)+\frac{1}{\Gamma(\gamma)}\int_0^{x_1}(x_1-s)^{\gamma-1}g(s,r)ds$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_1}(x_1-s)^{\gamma-1}a(s)w(s,r)ds$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_1}(x_1-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds)$$

$$\le \frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}g(s,r)ds,\int_0^{x_1}(x_1-s)^{\gamma-1}g(s,r)ds)$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}a(s)w(s,r)ds,$$

$$\int_0^{x_1}(x_1-s)^{\gamma-1}a(s)w(s,r)ds)$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds,$$

$$\int_0^{x_1}(x_1-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds)$$

$$\le \frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}g(s,r)ds+\int_{x_1}^0(x_2-s)^{\gamma-1}g(s,r)ds,$$

$$\int_0^{x_1}(x_1-s)^{\gamma-1}g(s,r)ds+\int_{x_1}^0(x_2-s)^{\gamma-1}g(s,r)ds)$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}a(s)w(s,r)ds$$

$$+\int_{x_1}^0(x_2-s)^{\gamma-1}a(s)w(s,r)ds,$$

$$\int_0^{x_1}(x_1-s)^{\gamma-1}a(s)w(s,r)ds+\int_{x_1}^0(x_2-s)^{\gamma-1}a(s)w(s,r)ds)$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_0^{x_2}(x_2-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds$$

$$+\int_{x_1}^0(x_2-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds,$$

$$\int_0^{x_1}(x_1-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds$$

$$+\int_{x_1}^0(x_2-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds)$$

$$\le \frac{1}{\Gamma(\gamma)}D(\int_{x_1}^{x_2}(x_2-s)^{\gamma-1}g(s,r)ds,$$

$$\int_0^{x_1}[(x_1-s)^{\gamma-1}-(x_2-s)^{\gamma-1}]g(s,r)ds)$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_{x_1}^{x_2}(x_2-s)^{\gamma-1}a(s)w(s,r)ds,$$

$$\int_0^{x_1}[(x_1-s)^{\gamma-1}-(x_2-s)^{\gamma-1}]a(s)w(s,r)ds)$$

$$+\frac{1}{\Gamma(\gamma)}D(\int_{x_1}^{x_2}(x_2-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$

$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds,$$

$$\int_0^{x_1}[(x_1-s)^{\gamma-1}-(x_2-s)^{\gamma-1}](\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$
$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds)$$
$$:= I_1+I_2+I_3.$$

where

$$I_1 \le \frac{1}{\Gamma(\gamma)}D(\int_{x_1}^{x_2}(x_2-s)^{\gamma-1}g(s,r)ds,\tilde{0})$$
$$+D(\int_0^{x_1}[(x_1-s)^{\gamma-1}-(x_2-s)^{\gamma-1}]g(s,r)ds,\tilde{0})$$
$$\le \frac{2(x_2-x_1)^{\gamma}}{\Gamma(\gamma+1)}\sigma+\frac{x_1^{\gamma}}{\Gamma(\gamma+1)}\sigma-\frac{x_2^{\gamma}}{\Gamma(\gamma+1)}\sigma$$
$$\le \frac{\sigma}{\Gamma(\gamma+1)}2(x_2-x_1)^{\gamma}.$$
$$I_2 \le \frac{1}{\Gamma(\gamma)}D(\int_{x_1}^{x_2}(x_2-s)^{\gamma-1}a(s)w(s,r)ds,\tilde{0})$$
$$+D(\int_0^{x_1}[(x_1-s)^{\gamma-1}-(x_2-s)^{\gamma-1}]a(s)w(s,r)ds,\tilde{0})$$
$$\le \frac{\zeta\lambda}{\Gamma(\gamma+1)}[2(x_2-x_1)^{\gamma}+x_1^{\gamma}-x_2^{\gamma}]$$
$$\le \frac{\zeta\lambda}{\Gamma(\gamma+1)}2(x_2-x_1)^{\gamma}.$$
$$I_3 \le \frac{1}{\Gamma(\gamma)}D(\int_{x_1}^{x_2}(x_2-s)^{\gamma-1}(\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$
$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds,\tilde{0})$$
$$+D(\int_0^{x_1}[(x_1-s)^{\gamma-1}-(x_2-s)^{\gamma-1}](\int_0^s k_1(s,\tau)F_1(w(\tau,r))d\tau$$
$$+\int_0^1 k_2(s,\tau)F_2(w(\tau,r))d\tau)ds,\tilde{0})$$
$$\le \frac{\sigma+\zeta\lambda+P_1\mu_1+P_2\mu_2}{\Gamma(\gamma+1)}2(x_2-x_1)^{\gamma}.$$

From I_1-I_3, when $x_1\to x_2$, it deduces

$$D(Tw(x_2,r),Tw(x_1,r))\to 0, \tag{4.8}$$

thus the equicontinuous result of T is obtained.

From the results of the previous steps (i), (ii) and (iii) and the Arzela-Ascoli Theorem, it can be concluded that T is a completely continuous operator.

(iv) Finally, we need to prove that there exists a closed convex bounded subset $B_\lambda=\{w(x,r)\in E: D(w(x,r),\tilde{0})\le\lambda\}$ such that $TB_\lambda\subseteq B_\lambda$,For each positive integer λ , $\exists w_\lambda\in B_\lambda$ such that $Tw_\lambda\notin TB_\lambda$,i.e $D(Tw_\lambda(x,r),\tilde{0})>\lambda$.

According to the proof of Eq.(4.6), it can be obtained

$$\lambda<\frac{1}{\Gamma(\gamma+1)}(\sigma+\zeta\lambda+\mu_1P+\mu_2P)+D(w(0,r),\tilde{0}), \tag{4.9}$$

Divide both sides of Eq.(4.9) by $\tilde{\lambda}$ to get

$$1<\frac{1}{\Gamma(\gamma+1)\lambda}(\sigma+\zeta\lambda+\mu_1P+\mu_2P)+\frac{D(w(0,r),\tilde{0})}{\lambda}, \tag{4.10}$$

When $\lambda\to\infty$, it deduces

$$1<\frac{\zeta}{\Gamma(\gamma+1)}, \tag{4.11}$$

It is easy to get the contradiction.

Through the proof of (i)-(iv), we know that T has at least one fixed point u(x) on CF [0, 1], such that u(x) is the solution of Eqs.(1.3)-(1.4).

5. Numerical results

In this section, we will use ADM to solve fuzzy nonlinear fractional Volterra-Fredholm integro-differential equations. The iterative sequence of the approximate solution is given in section 3. For the Adomian polynomial corresponding to the nonlinear term, see [41].

Example 1 Consider the following fuzzy nonlinear fractional Volterra-Fredholm integro-differential equations defined in the Caputo sense

$$\begin{cases}\begin{cases} {}^cD^{1/2}[\underline{w}(x,r)]=\frac{rx^{1/2}}{5\Gamma(1/2)}-\frac{r^2x^2}{300}-\frac{rx^4}{40}\underline{w}(x,r)\\ +\int_0^x xt\underline{w}^2(t,r)dt+\int_0^1 x^2\underline{w}^2(t,r)dt,\\ \underline{w}(0,r)=0,\end{cases}\\ \begin{cases} {}^cD^{1/2}[\overline{w}(x,r)]=\frac{(2-r)x^{1/2}}{5\Gamma(1/2)}-\frac{(2-r)^2x^2}{300}\\ -\frac{(2-r)x^4}{40}\overline{w}(x,r)+\int_0^x xt\overline{w}^2(t,r)dt+\int_0^1 x^2\overline{w}^2(t,r)dt,\\ \overline{w}(0,r)=0.\end{cases}\end{cases}$$

The exact solution is

$$w(x,r)=[\underline{w}(x,r),\overline{w}(x,r)]=[\frac{r}{10}x,\frac{2-r}{10}x],r\in[0,1]$$

It can be written from the recursive

$$\underline{w}_0(x,r)=\underline{w}(0,r)+J^{1/2}(\frac{rx^{1/2}}{5\Gamma(1/2)}-\frac{r^2x^2}{300}),$$
$$\underline{w}_{i+1}(x,r)=J^{1/2}(-\frac{rx^4}{40}\underline{w}_i(x,r)$$
$$+\int_0^x xt\underline{A}_i(t,r)dt+\int_0^1 x^2\underline{B}_i(t,r)dt),\ i=0,1,2\cdots.$$
$$\overline{w}_0(x,r)=\overline{w}(0,r)+J^{1/2}(\frac{(2-r)x^{1/2}}{5\Gamma(1/2)}-\frac{(2-r)^2x^2}{300}),$$
$$\overline{w}_{i+1}(x,r)=J^{1/2}(-\frac{(2-r)x^4}{40}\overline{w}_i(x,r)$$
$$+\int_0^x xt\overline{A}_i(t,r)dt+\int_0^1 x^2\overline{B}_i(t,r)dt),\ i=0,1,2\cdots.$$

Select $r=0.4, n=2$. The left and right boundary errors between the approximate solution and the exact solution are obtained in Table 1 and Table 2.

Table1: left bound of error

x	exact	ADM	**error**
0	0	0	0
0.2	8.0000e-03	7.9999e-03	6.1249e-08
0.4	1.6000e-02	1.6000e-02	3.4732e-07
0.6	2.4000e-02	2.3999e-02	9.6977e-07
0.8	3.2000e-02	3.1998e-02	2.0726e-06
1.0	4.0000e-02	3.9996e-02	3.9602e-06

Table2: right left bound of error

x	exact	ADM	**error**
0	0	0	0
0.2	3.2000e-02	3.1996e-02	3.8844e-06
0.4	6.4000e-02	6.3978e-02	2.2028e-05
0.6	9.6000e-02	9.5939e-02	6.1500e-05
0.8	1.2800e-01	1.2787e-01	1.3137e-04
1.0	1.6000e-01	1.5975e-01	2.5052e-04

Example 2 Solve the following fuzzy Caputo fractional Volterra-Fredholm integro-differential equation

$$\begin{cases}\begin{cases} {}^cD^{1/2}[\underline{w}(x,r)] = \dfrac{rx^{1/2}}{6\Gamma(1/2)} - \dfrac{rx^3}{72} - \dfrac{x^4}{3}\underline{w}(x,r) \\ +\int_0^x x^2 t\underline{w}(t,r)dt + \int_0^1 x^3(1-t)\underline{w}(t,r)dt, \\ \underline{w}(0,r)=0, \end{cases} \\ \begin{cases} {}^cD^{1/2}[\overline{w}(x,r)] = \dfrac{(2-r)x^{1/2}}{6\Gamma(1/2)} - \dfrac{(2-r)x^3}{72} - \dfrac{x^4}{3}\overline{w}(x,r) \\ +\int_0^x x^2 t\overline{w}(t,r)dt + \int_0^1 x^3(1-t)\overline{w}(t,r)dt, \\ \overline{w}(0,r)=0. \end{cases}\end{cases}$$

The exact solution is

$$w(x,r) = [\underline{w}(x,r), \overline{w}(x,r)] = [\frac{r}{15}x, \frac{2-r}{15}x], r \in [0,1]$$

It can be written from the recursive

$$\underline{w}_0(x,r) = \underline{w}(0,r) + J^{1/2}(\frac{2rx^{1/2}}{15\Gamma(1/2)} - \frac{r^3}{16875}),$$

$$\underline{w}_{i+1}(x,r) = J^{1/2}(-\frac{rx^4}{75}\underline{w}_i(x,r)$$

$$+\int_0^x t^2 \underline{A}_i(t,r)dt + \int_0^1 t\underline{B}_i(t,r)dt),\ i = 0,1,2\cdots.$$

$$\overline{w}_0(x,r) = \overline{w}(0,r) + J^{1/2}(\frac{2(2-r)x^{1/2}}{15\Gamma(1/2)} - \frac{(2-r)^3}{16875}),$$

$$\overline{w}_{i+1}(x,r) = J^{1/2}(-\frac{(2-r)x^4}{75}\overline{w}_i(x,r)$$

$$+\int_0^x t^2 \overline{A}_i(t,r)dt + \int_0^1 t\overline{B}_i(t,r)dt),\ i = 0,1,2\cdots.$$

Select $r = 0.6, n = 2$. The left and right boundary errors between the approximate solution and the exact solution are obtained in Table 31 and Table 4.

Table3: left bound of error

x	**exact**	ADM	**error**
0	0	0	0
0.2	8.0000e-03	8.0000e-03	7.7909e-09
0.4	1.6000e-02	1.6000e-02	1.1621e-08
0.6	2.4000e-02	2.4000e-02	1.8248e-08
0.8	3.2000e-02	3.2000e-02	3.5719e-08
1.0	4.0000e-02	4.0000e-02	7.8953e-08

Table4: right left bound of error

x	**exact**	**ADM**	**error**
0	0	0	0
0.2	1.8667e-02	1.8666e-02	5.3710e-07
0.4	3.7333e-02	3.7333e-02	7.7739e-07
0.6	5.6000e-02	5.5999e-02	1.0709e-06
0.8	7.4667e-02	7.4665e-02	1.6700e-06
1.0	9.3333e-02	9.3330e-02	3.0218e-06

Through the analysis of error results in Table 1-Table 4, it can be found that the error results are ideal and the error is small, which shows the effectiveness and practicability of ADM.

6. Conclusions

In this paper, the existence of solutions for fuzzy nonlinear fractional Volterra-Fredholm integro-differential equations are proved by using Schauder fixed point theorem. The nonlinear part of the equation is approximated by Adomian polynomials, and then the iterative sequence of the numerical solution of the equation is constructed. Finally, the fast convergence of ADM is verified by the results of numerical examples, which also shows the effectiveness and applicability of the method.

References

[1] N. H. Abel. Solution de quelques problemes a láide díntegrales definites. Chris- tiania Grondahl, Norway, 1(1881), 16-18.

[2] R. P. Agarwal, V. Lakshmikantham, J. J. Nieto. On the concept of solution for fractional differential equations with uncertainty. Nonlinear Anal, 72(2010), 59-62.

[3] M. Rahaman, S. P. Monda, A. A. Shaikh, A. Ahmadian, N. Senu, S. Salahshour. Arbitrary-order economic production quantity model with and without deerioration:generalized point of view. Advances in Difference Equations, 2020(2020), 1-30.

[4] S. S. L. Chang, L. A. Zadeh. On fuzzy mapping and control. IEEE Transactions on Systems, Man, and Cybernetics, 2(1972), 30-34.

[5] L. A. Zadeh. The concept of linguistic variable and its application to approximate reasoning. Information Sciences, 8(1975), 199-249.

[6] M. Mizumoto, K. Tanaka. The four operations of arithmetic on fuzzy numbers. Systems Comput Controls, 7(1976), 73-81.

[7] D. Dubois, H. Prade. Towards fuzzy differential calculus, Part I: integration of fuzzy mappings, class of second-order. Fuzzy Sets and Systems, 8(1982), 1-17.

[8] M. Puri, D. Ralescu. Differentials of fuzzy functions. Journal of Mathematical Analysis and Applications, 91(1983), 552-558.

[9] B. Bede, S. G. Gal. Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equations. Fuzzy Sets and Systems, 151(2005), 581-599.

[10] L. Stefanini, B. Bede. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. Nonlinear Analysis, 71(2009), 1311-1328.

[11] S. Alkan, V. Hatipoglu. Approximate solutions of Volterra-Fredholm integro- differential equations of fractional order. Tbilisi Mathematical Journal, 10(2017), 1-13.

[12] A. A. Hamoud, A. D. Azeez, K. P. Ghadle. A study of some iterative methods for solving fuzzy Volterra-Fredholm integral

equations. Indonesian Jour- nal of Electrical Engineering and Computer Science, 11(2018), 1228-1235.

[13] A. A. Hamoud, K. P. Ghadle. The reliable modified of Laplace Adomian decomposition method to solve nonlinear interval Volterra-Fredholm integral equations. The Korean Journal of Mathematics, 25(2017), 323-334.

[14] A. A. Hamoud, K. P. Ghadle. The combined modified Laplace with Adomian decomposition method for solving the nonlinear Volterra-Fredholm integro-differential equations. Journal of the Korean Society for Industrial and Applied Mathematics, 21(2017), 17-28.

[15] X. Ma, C. Huang. Numerical solution of fractional integro-differential equations by a hybrid collocation method. Applied Mathematics and Computation, 219(2013), 6750-6760.

[16] R. Mittal, R. Nigam. Solution of fractional integro-differential equations by Adomian decomposition method. International Journal of Advances in Applied Mathematics and Mechanics, 4(2008), 87-94.

[17] S. Arshad, V. Lupulescu. On the fractional differential equations with uncertainty. Nonlinear Anal, 74(2011), 85-93.

[18] S. Arshad, V. Luplescu. Fractional differential equation with fuzzy initial conditon. Electronic Journal of Differential Equations, 34(2011), 1-8.

[19] T. Allahviranloo, A. Armand, Z. Gouyandeh, H. Ghadiri. Existence and uniqueness of solutions for fuzzy fractional Volterra-Fredholm integro-differential equations. Journal of Fuzzy Set Valued Analysis, 2013(2013), 1-9.

[20] N. Ahmad, A. Ullah, A. Ullah, S. Ahmad, K. Shah, I. Ahmad. On analysis of the fuzzy fractional order Volterra-Fredholm integro-differential equation. Alexandria Engineering Journal, 60(2020), 1827-1838.

[21] A. A. Hamoud, K. P. Ghadle. Modified Laplace decomposition method for fractional Volterra-Fredholm integro-differential equations. Journal of Mathe- matical Modeling, 6(2018), 91-104.

[22] A. A. Hamoud, K. P. Ghadle, S. M. Atshan. The approximate solutions of fractional integro-differential equations by using modified Adomian decomposition method. Khayyam Journal of Mathematics, 5(2019), 21-39.

[23] A. A. Hamoud, K. P. Ghadle. Homotopy analysis method for the first order fuzzy Volterra-Fredholm integro-differential equations. Indonesian Jour- nal of Electrical Engineering and Computer Science, 11(2018), 857-867.

[24] M. S. Bani Issa, A. A. Hamoud, K. P. Ghadle. Numerical solutions of fuzzy integro-differential equations of the second kind. Journal of Mathematics and Computer Science, 23(2021), 67-74.

[25] M. Baghmisheh, R. Ezzati. Numerical solution of nonlinear fuzzy Fred- holm integral equations of the second kind using hybrid of block-pulse functions and Taylor series. Advances in Difference Equations, 2015(2015), 1-15.

[26] S. Salahshour, A. Ahmadian, N. Senu, D. Baleanu, P. Agarwal. On analytical solutions of the fractional differential equation with uncertinity: Application to the basset problem. Entropy, 17(2015), 855-902.

[27] D. Dubois, H. Prade. Operations on fuzzy numbers. In-ternational Journal of Systems Science, 9(1978), 613-626.

[28] S. S. Behzadi, T. Allahviranloo, S. Abbasbandy. Solving fuzzy second-order nonlinear Volterra-Fredholm integro-differential equations by using Picard method. Neural Computing and Applications, 21(2012), 337-346.

[29] W. Al-Hayani. Solving fuzzy system of Volterra integro-differential equations by using Adomian decomposition method. European Journal of Pure and Applied Mathematics, 15(2022), 290-213.

[30] S. Seikkala. On the fuzzy initial value problem. Fuzzy Sets and Systems, 24(1987), 319-330.

[31] S. G. Gal. Approximation theory in fuzzy setting. Handbook of analyticcomputational methods in applied mathematics. Chapman and Hall/CRC Press, Boca Raton, 2019, 3-50.

[32] W. Feng, D. Zhang. The local existence and uniqueness of solutions for fuzzy functional Volterra integral equations. 2012 9th International Conference on Fuzzy Systems and Knowledge Discovery, 2012, 184-187.

[33] S. Hajighasemi. Fuzzy Fredholm-Volterra integral equations and existance and uniqueness of solution of them. Australian Journal of Basice and Applied Sci- ence, 5(2011), 1-8.

[34] [34] B. Bede, L. Stefanini. Generalized differentiability of fuzzy-valued functions. Fuzzy Sets and Systems, In press, 230(2013), 119-141.

[35] T. Allahviranloo, S. Salahshour, S. Abbasbandy. Explicit solutions of fractional differential equations with uncertainty. Soft Comput, 16(2012), 297-302.

[36] S. Salahshour, T. Allahviranloo S. Abbasbandy. Solving fuzzy frac- tional differential equations by fuzzy Laplace transforms. Communications in Nonlinear Science and Numerical Simulation, 17(2012), 1372-1381.

[37] A. M. Wazwaz. The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations. Applied Mathematics and Computation, 216(2010), 1304-1309.

[38] K. Abbaoui, Y. Cherruault. Convergence of Adomian s method applied to nonlinear equations. Mathematical and Computer Modelling, 20(1994), 69-73.

[39] G. Adomian. A review of the decomposition method in applied mathematics. Journal of Mathematical Analysis and Applications, 135(1988),501-544.

[40] A. M. Wazwaz. A reliable modification of Adomian decomposition method. Applied Mathematics and Computation, 102(1999), 77-86.

[41] M. O. Olayiwola, K. O. Kareem. Efficient decomposition method for integrodifferential equations. Journal of Mathematics and Computer Science, 12(2022), 66-81.