

Philosophical Justifications for the Axiom of Choice

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Abstract: This paper investigates two kinds of philosophical justifications—intrinsic and extrinsic justifications—for the Axiom of Choice. A standard intrinsic justification for the Axiom of Choice is based on the iterative conception of set. However, this form of justification implicitly adopts the maximality principle of sets, which is incoherent. The extrinsic justification turns out to be more effective due to the theoretical merits of the Axiom of Choice, despite the paradoxes caused by AC.

Keywords: The Axiom of Choice; ZF Set Theory; Intrinsic and extrinsic justification.

1. Introduction

Set theory is a branch of mathematical logic that studies sets, which has long been regarded as the foundation of mathematics as it provides a universal language to describe mathematical objects. The standard set theory is Zermelo-Fraenkel (ZF) set theory, in which nearly all mathematical objects such as numbers and functions can all be represented in set theory. Set theory also enables a rigorous study of infinity. Infinities of different sizes and types can not only be defined but also be compared within the scope of set theory.

As mentioned above, ZF set theory is the most accepted set theory, which includes the following axioms:

- Axiom of Extensionality: $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$.
- Axiom of Separation: $\forall x \exists y \forall v (v \in y \leftrightarrow v \in x \wedge \varphi(v))$.
- Axiom of Pairing: $\forall x \forall y \exists z \forall v (v \in z \leftrightarrow v = x \vee v = y)$.
- Axiom of Union: $\forall x \exists y \forall z (z \in y \leftrightarrow \exists v (v \in x \wedge z \in v))$.
- Axiom of Power Set: $\forall x \exists y \forall z (z \in y \leftrightarrow \forall v (v \in z \rightarrow v \in x))$.
- Axiom of Foundation: $\forall x (\exists y (y \in x) \rightarrow \exists z (z \in x \wedge \neg \exists w (w \in z \wedge w \in x)))$.
- Axiom of Infinity: $\exists x (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))$.
- Axiom of Replacement: $\forall x [\forall u \in x \exists! v \varphi(u, v) \rightarrow \exists y \forall v (v \in y \leftrightarrow \exists u (u \in x \wedge \varphi(u, v)))]$.

In addition to the ZF axioms, another critical axiom is the Axiom of Choice (AC). The combination of ZF set theory and the axiom of choice is called ZFC. To start with, a function f is said to be a choice function on a set X , if the domain of f is X and for every $x \in X$, $f(x) \in x$. The Axiom of Choice states that every collection of non-empty sets has a choice function.

In comparison to the other axioms of ZF set theory, AC is more controversial in mathematics. On the one hand, standard philosophical justifications for AC face objections. On the other hand, AC leads to paradoxical and counterintuitive results. This paper argues that the standard philosophical conception of sets is not capable of offering an effective justification for AC. However, it is shown that AC can still be

justified from another perspective.

The structure of this paper is as follows. Section 2 provides a more detailed introduction to AC as a special axiom and distinguishes two kinds of philosophical justifications. Section 3 discusses the intrinsic justification of AC from a philosophical conception of set theory. This justification, however, is considered invalid because of the implicit adoption of an inconsistent principle. Section 4 discusses how AC is extrinsically justified based on its application to algebra and analysis. Finally, it is concluded that AC can be justified extrinsically but not intrinsically.

2. Preliminaries

AC has both theoretical merits and paradoxical consequences, which makes it the most special axiom. AC is equivalent to several foundational theorems, such as Zorn's Lemma, which are widely utilized in different mathematical areas including abstract algebra and topology. Thus, since accepting the axiom is the same as accepting any of its equivalent forms, AC is critical to many fields of mathematics. Accepting AC, on the other hand, will also bring some defects since it leads to some paradoxical consequences. This point will be further illustrated in Section 4.

Another interesting aspect of AC is its independence from ZF. Specifically, Gödel (1940) proved that if ZF is consistent, ZFC is consistent. However, it is also proved by Cohen (1963) that if ZF is consistent, then ZF plus $\neg$ AC is also consistent. That is, ZF cannot decide the truth of AC.

Due to this independence phenomenon, the justifications of AC must be philosophical. There are two types of justifications for axioms in set theory: intrinsic and extrinsic justifications (Gödel 1947; Maddy 1988; Barton 2022). An axiom of set theory is intrinsically justified if it conforms to a coherent philosophical conception of sets. For example, the Axiom of Extensionality can be intrinsically justified, because it is commonly held that sets are determined by their elements. By contrast, extrinsic justifications mostly concern the usefulness, applicability, or explanatory power of the axiom in mathematics. This kind of justification is similar to the justifications for the fundamental principles in physics. For example, in kinematics, a formula is more likely to be accepted if it accurately and concisely predicts the motion of an object. In the rest of the paper, we will discuss how AC can be justified from these two perspectives.

3. Intrinsic Justification

The first theory people establish to study sets is the Naïve Comprehension Axiom, asserting that, for any property $P(x)$, there exists a set S such that for any element x , x is an element of S if and only if $P(x)$ holds. However, this theory is proved to be inconsistent due to Russell's Paradox (1901), which is the simple observation that $R = \{x | x \notin x\}$ cannot be a set. This is primarily because if an element belongs to R , the element will not belong to it because this element does not satisfy the property $x \notin x$. For the same reason, if an element does not belong to R , then the element should belong to R , because the element has satisfied the property $x \notin x$. Therefore, according to the Naïve Comprehension Axiom, it can be derived that an element belongs to R if and only if it does not belong to R , which is a contradiction. The iterative conception is therefore suggested as an alternative perspective to understand the construction of sets, aiming to resolve Russell's Paradox. The iterative view suggests that sets are formed in the following way.

Sets are formed in stages. For each stage S , there are certain stages which are before S . At stage S , each collection consisting of sets formed at stages before S is formed into a set. There are no sets other than the sets which are formed at stages. (Shoenfield, 1977, p. 323)

In other words, the iterative conception proposes two principles:

Generation principle: Things that have already appeared in each stage will form their sets in the next stage.

Restriction principle: Every set appears at some stage.

Thus, the iterative conception describes a hierarchy of sets as demonstrated below. At the first stage, nothing exists, so there's only the empty set. On the second stage, the only thing that can be used below that stage is the empty set. As a result, the second stage contains the singleton of empty set and empty set itself. So forth and so on.

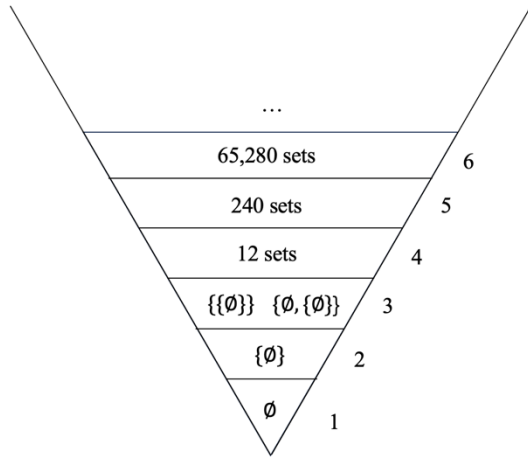


Figure 3.1, Illustration of the iterative conception

The iterative conception resolves Russell's Paradox in the following way. Consider $R_S = \{x | x \notin x \text{ and } x \text{ was available before } S\}$, where S is a stage. First, R_S is not an element of itself, because if R_S is, Russell's Paradox will be triggered as described before. Second, if the Russell class $R = \{x | x \notin x\}$ occurs in a certain stage S , then R is R_S . This is because only the elements x exist before stage S can be used to compose R , which is equivalent to the definition of R_S . Therefore, R cannot appear in any of the stages. According to the Restriction Principle, since the Russell class doesn't exist at

any stage, a Russell class is not a set.

ZF set theory is based on the iterative conception. In fact, most of the axioms in ZF set theory are verified to be conforming to the iterative conception. Take the Axiom of Union as an example. The Axiom of Union states that for any set A , there exists a set $\cup A$ such that its elements are precisely those elements of each element in A . This conforms with the iterative conception since the elements of the elements in A must appear before the stage where A appears, and so $\cup A$ will form a set on the next stage.

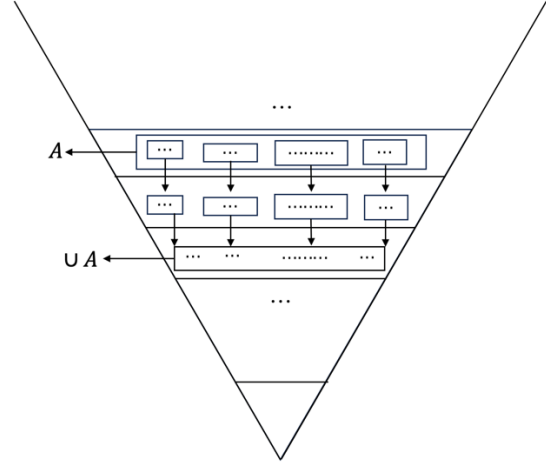


Figure 3.2, Illustration of the Axiom of Union

Therefore, the most natural intrinsic justification for AC should be based on the iterative conception. Let us now turn to a standard argument that aims to derive AC from the iterative conception of set.

To start with, the following statement is equivalent to AC:

(AC*) If the members of A are disjoint and non-empty, then A has a choice set C such that $C \cap x$ is a singleton for every $x \in A$.

Proof:

We first show that AC implies (AC*).

Let A be a family of disjoint, non-empty sets. By AC, we can choose one element i from each set in the family A_i . Specifically, AC guarantees the existence of a choice function f such that $f(i) \in A$ for each i . Let $C = \text{ran}(f)$. For each i , the set $A_i \cap C$ contains exactly one element, namely $f(i)$. Thus, $A_i \cap C = \{f(i)\}$, which is a singleton. This is the proof of AC implies (AC*).

Next, we show that (AC*) implies (AC). Let $B = \{B_i | i \in I\}$ be a family of non-empty sets. We need to show that there exists a choice function f such that $f(B_i) \in B_i$ for each i . Now, consider the set $A = \{\{B_i\} \times B_i | i \in I\}$. Each element of A is a disjoint non-empty set because i is unique to each set. Then, apply (AC*) to A . By (AC*), there exists a set C such that for each i , $(B_i \times i) \cap C$ is a singleton. Let $f = \{(B_i, x) | (B_i, x) \in C\}$. It clears that f is a choice function on B . Therefore, AC is equivalent to (AC*). Q.E.D

Given the equivalence between AC and (AC*), if (AC*) can be justified by the iterative conception, then so is AC. According to the iterative conception, A is formed at some stage S and all elements in $\cup A$ are available before S . Therefore, it is reasonable to argue that, since every element to form a choice set is available before S , the choice set consisting of some of these elements will appear at S , which is a subset of A . (Button, 2021)

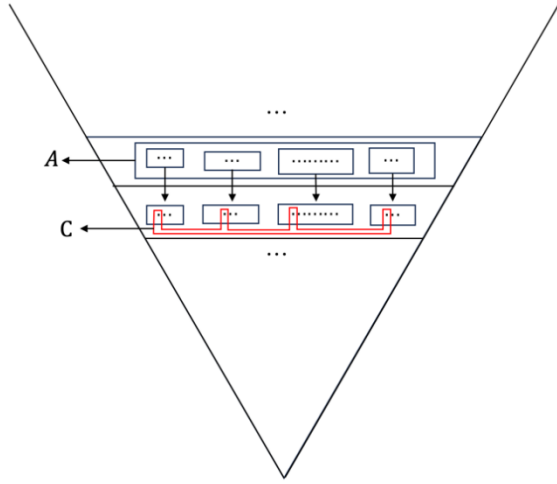


Figure 3.3, Illustration of the Axiom of Choice

However, this argument faces objections. The last step of inference implicitly adopts the view that whatever can exist does exist in the set-theoretic universe. Let's call this view the maximality principle. Yet, this version of the maximality principle is incoherent.

On the one hand, the maximality principle supports the negation of the Continuum Hypothesis (CH). CH asserts that there is no cardinal number between countable infinity and the continuum. Since Cohen (1963) famously showed that the negation CH is consistent with ZF, the existence of such cardinals is possible. So according to the maximality principle, cardinal numbers exist because whatever can exist does exist. Therefore, adopting the maximality principle means CH is false. On the other hand, the maximality principle can also support CH. An equivalent form of CH is "There is a bijection between a continuum and countable infinity". The existence of this bijection is shown to be possible since Gödel (1940) showed that CH is consistent with ZF. Again, according to the maximality principle, this bijection exists. Since the maximality principle supports both CH and the negation of CH, the maximality principle is inconsistent.

The second consequence is a contradiction between the Axiom of Determinacy (AD) and AC, which are two competing theories. AD states the following: Consider a two-person infinite game in which Player I and II take turns choosing a number from the set of natural numbers $\mathbb{N}$ to form an infinite sequence $(a_0, a_1, a_2, \dots)$. Let A be any set of natural number sequences. If player I can manage to make the resulting sequence belong to set A , then player I wins. If not, Player II wins. AD asserts that for any given set A of natural number sequences, the game with respect to A is determined. In other words, one player has a strategy that ensures that he wins the game no matter what the other player does. This strategy is called a winning strategy. The maximality principle also supports AD, because the maximality principle expects all possible infinite games to have certain winning strategies. However, AD and AC contradict. If AC is also supported by the maximality principle, then a contradiction will be derived from the maximality principle. In other words, the maximality principle can't support AC since it is inconsistent.

In summary, the iterative conception does not provide an effective intrinsic justification for AC, because the maximality principle is utilized in the justification and is proven to be inconsistent.

4. Extrinsic Justification

The Axiom of Choice has many theoretical merits. To begin with, some of the equivalents of AC are widely utilized in other fields of mathematics.

In linear algebra, it is intuitively acceptable that every vector space has a basis. This provides a foundation for understanding the structure and properties of vector spaces and simplifies problem-solving. However, it is AC that guarantees this property. It is proven by Banach and Tarski (1924) that AC implies "every vector space has a basis". Blass (1984) proved that "every vector space has a basis" implies AC, hence they are equivalent. In other words, discarding AC will lead to the existence of vector spaces without a basis. Since every vector in a linear space is a linear combination of the basis, the vector space without a basis will contain no vector in it, which is a highly pathological result in linear algebra.

As for analysis, AC is used in proving the equivalence of two definitions of continuity of functions: A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a point x if:

$\forall \varepsilon > 0, \exists \delta > 0$ such that $\forall y$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. (Definition 1)

$\lim_{n \rightarrow \infty} x_n = x \Rightarrow \lim_{n \rightarrow \infty} f(x_n) = f(x)$. (Definition 2)

The first definition is the Cauchy definition while the second one is the pointwise limit definition. These two definitions are used in different situations. In the context of calculus, the first definition turns out to be more practical when dealing with derivatives and integrations, whereas the second definition outweighs the first one when coping with the convergence and divergence of series. Naturally, these two definitions are expected to be equivalent in the context of mathematics. But this equivalence also relies on AC. If we deny AC, the foundation of mathematical analysis will be fractured. Therefore, we may conclude that AC is extrinsically justified due to its mathematical consequences.

However, introducing AC begets paradoxical and counterintuitive results.

Lebesgue measure is a natural method to depict the magnitude of sets of real numbers. One of the most undesirable consequences AC brought is the existence of non-Lebesgue measurable sets, which are pathological (Vitali, 1904). The pathological result of non-Lebesgue measurable sets further generates the Banach-Tarski Paradox, proved by Banach and Tarski (1924), which states that one can decompose a three-dimensional ball into two balls that are all equal in size to the original one. A decomposition as such is clearly against our geometric intuition.

However, it is commonly held that the theoretical merits of AC outweigh its paradoxical consequences, simply because AC plays many indispensable roles in different areas of mathematics, as we have seen. Therefore, it is reasonable to conclude that AC is well-justified extrinsically.

5. Conclusion

In summary, we argue that the iterative conception, as a commonly held philosophical principle, fails to provide an effective intrinsic justification for AC. This is because justifying AC based on the iterative conception inevitably adopts the maximality principle, which, shown with two arguments, is inconsistent. However, the merits brought by AC, including the fundamental theorems in different fields of mathematics, significantly outweigh the drawbacks.

Therefore, we conclude that AC can be extrinsically justified.

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