

Myerson's Lemma under Fuzzy Valuation

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Abstract: The paper explores extending Myerson's lemma, a key concept in mechanism design, to situations where participants' valuations are imprecise. Traditional game theory assumes precise valuations, but real-world scenarios often involve uncertainties. By introducing fuzzy set theory, the paper addresses this issue, allowing valuations to be represented as fuzzy numbers. The study proves that even under fuzzy valuations, the auction mechanism can still satisfy dominant strategy incentive compatibility (DSIC). Practical applications are discussed, including auctions in online advertising where click-through rates (CTR) are uncertain. The paper demonstrates how fuzzy valuation can ensure fairness and efficiency in these auctions, offering a more realistic approach to mechanism design in uncertain environments.

Keywords: Fuzzy valuation; Mechanism design; Dominant strategy incentive compatibility (DSIC).

1. Introduction

Algorithmic game theory, a convergence of game theory and computer science, has a rich history and a promising future. Its potential to solve complex decision-making problems, from resource distribution to auction mechanism design, is vast. The field began to take shape in the mid-twentieth century, when seminal works like *Theory of Games and Economic Behavior* (1944) [1] laid the systematic foundations of game theory, coinciding with the rapid development of computer science. This theory has many applications in real life. For example, power producers will use algorithmic game theory to bid since the bid directly links to the market shares and profit per charge unit. Additionally, because the electricity market is relatively unstable and others' biddings are closed, algorithmic game theory can give producers a general guideline for bidding and producing precisely instead of analyzing cases by case.[5]

We presume that the participants in algorithmic game theory are rational, which means they assign a point value to specific services. However, in real life, participants usually face many indeterminacies caused by incomplete information, the subjectivity of evaluation, etc. To describe and deal with this uncertainty more accurately, fuzzy mathematics and fuzzy set theory are introduced into the study of algorithmic game theory. Fuzzy set theory, proposed by Zadeh in 1965, provides a powerful tool for the mathematical description of fuzzy concepts by introducing the idea of membership degree. [2] Introducing this theory not only makes the algorithmic game theory better cope with the fuzziness in the real world but also significantly improves the practicability and accuracy of the model, instilling confidence and trust in its application.

Against this background, introducing fuzzy set theory into Myerson's lemma in algorithmic game theory becomes an important research direction. As one of the core theorems in mechanism design theory, Myerson's lemma provides solid theoretical support for designing auction mechanisms that satisfy the properties of incentive compatibility and individual rationality. Because of this lemma, people can only consider the monotony of assigning rules instead of building a complex system to fulfill DSIC. Therefore, studying Myerson's lemma under fuzzy valuation can expand the application scope of game theory. Besides, it provides new

ideas and methods for designing a more realistic auction mechanism and resource allocation algorithm.

For instance, the auction mechanism is critical for trading commodities in electronic commerce. By introducing Myerson's lemma under fuzzy valuation, the platform can design a fairer and more transparent auction mechanism, reduce information asymmetry and moral hazard in transactions, and improve transaction efficiency and user satisfaction.

This paper will discuss the Myerson lemma under fuzzy valuation and its application potential in depth. Reviewing the historical development of algorithmic game theory, it analyzes the reasons for introducing fuzzy numbers. At the same time, the necessary mathematical proof is introduced, and the concrete application case is illustrated. To sum up, this paper hopes to provide new ideas and methods for researchers in related fields.

2. Preliminaries

2.1. Game theory

Game theory uses rigorous mathematical models to study the optimal decision problem under conflict and confrontation conditions. Specifically, it analyzes how decision-making agents make decisions to maximize utility under a given information structure and discusses the equilibrium state between different decision-making agents. Two important concepts in game theory are Nash equilibrium and dominant strategy equilibrium. These foundational concepts were first introduced by Von Neumann and Morgenstern in their seminal work *Theory of Games and Economic Behavior* [1].

Definition 1.1 Nash Equilibrium: Nash equilibrium refers to a combination of strategies. In this combination, every player's strategy best reacts to the others' strategy [1].

Definition 1.2 Dominant Strategy Equilibrium: Dominant strategy equilibrium belongs to Nash Equilibrium. In this combination, every player chooses its strategy regardless of which others choose [1].

Mechanism Design: Mechanism design theory studies designing a reasonable mechanism (system or rule) to achieve the designer's established social goal or economic efficiency under the decentralized decision-making conditions of free choice and voluntary exchange. Incentive compatibility

(DSIC) is a crucial concept in mechanism design. The concept of DSIC is especially crucial in auction design, as introduced by Myerson in his work on optimal auction design [3].

Definition 1.3. Dominant Strategy Incentive Compatible: DSIC means the utility must be maximized by bidding truthfully for every bidder i [3].

2.2. Fuzzy Mathematics

Fuzzy numbers are generalizations of ordinary real numbers used to describe fuzzy values. Many values are unknown in the real world, but there is a certain ambiguity. Fuzzy numbers [2] describe this fuzziness by introducing the concept of membership degree.

To be specific, we represent an arbitrary fuzzy number by an ordered pair of functions $(\underline{u}(r), \bar{u}(r))$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$.
2. $\bar{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$.
3. $\underline{u}(r) \leq \bar{u}(r)$, $0 \leq r \leq 1$.

Here's the basic calculation of fuzzy numbers. For arbitrary fuzzy numbers $x = (\underline{x}(r), \bar{x}(r))$, $y = (\underline{y}(r), \bar{y}(r))$ and real number k , we have the following arithmetic operations:

- Equality: $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\bar{x}(r) = \bar{y}(r)$.
- Addition: $x + y = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$.
- Scalar Multiplication: $kx = (k\underline{x}, k\bar{x})$ for $k \geq 0$, and $(kx, k\bar{x})$ for $k < 0$. [4]

The following presents the triangular fuzzy number, a common fuzzy number type.

Triangular Fuzzy Number

A triangular fuzzy number is a triangular-shaped fuzzy number that is usually defined by three parameters: the left endpoint a , the middle point m , and the correct endpoint b ($a \leq m \leq b$). The membership function of the triangular fuzzy number increases linearly on the interval $[a, m]$, decreases linearly on the interval $[m, b]$, and reaches the maximum 1 at m . For instance, $A(1,2,4)$ will be like a triangle with three endpoints: $(1,0)$, $(2,1)$, $(4,0)$, and the dependent variable is membership degree.

3. Main Results

To begin with, Myerson's Lemma claims that for a single-parameter environment (i), an allocation rule x is implementable, which means there is a payment rule p that can make mechanism (x, p) become DSIC if and only if x is monotone; (ii) this p is unique if we assume if $b_i=0$, then

$$Z(r) \cdot x(z) - p(z) \geq Z(r) \cdot x(y) - p(y) \quad (1)$$

This ensures that bidding a higher value z results in at least the same utility as bidding y .

3. Conversely, to prevent bidders from gaining more by

$$y \cdot x(z) - p(z) \leq y \cdot x(y) - p(y) \quad (2)$$

This prevents any utility gain from underbidding, thus

$p_i(b) = 0$. At last, this p can be expressed clearly by the formula. To adjust this lemma to daily life, b_i (evaluation to the object from bidder i) is a triangular fuzzy number.

Definition 2.1: In this paper, $A(r)$ refers to a triangular fuzzy number. Other variables, such as those related to bids, will be represented using lowercase letters, such as $p(z)$, where $p(z)$ specifically refers to bid prices. The fuzzy numbers are represented by their membership functions, with $A(r)$ indicating a fuzzy number, rather than the lower or upper bounds directly.

Additionally, when comparing two fuzzy numbers $A(r)$ and $B(r)$, it is necessary to use their respective membership functions to establish a valid relationship. This comparison is usually done by comparing their α -cuts. Specifically, $A(r) \leq B(r)$ indicates that for any given $\alpha \in [0,1]$, the α -cut of $A(r)$, denoted as $[A_\alpha^-, A_\alpha^+]$, should be less than or equal to the α -cut of $B(r)$, denoted as $[B_\alpha^-, B_\alpha^+]$, meaning $A_\alpha^+ \leq B_\alpha^+$ and $A_\alpha^- \leq B_\alpha^-$.

Furthermore, comparing a constant a with a fuzzy number $B(r)$ requires a specific method to ensure the validity of the comparison. Typically, this is done by checking whether the constant a falls within the range of the α -cuts of $B(r)$ for all $\alpha \in [0,1]$. More formally, $a \leq B(r)$ means $a \leq B_\alpha^-$ for all α , and similarly, $a \geq B(r)$ means $a \geq B_\alpha^+$ for all α . This method accounts for the degree of overlap and ensures a consistent interpretation of the relationship between a crisp number and a fuzzy set.

This framework is applicable because in real-life scenarios, bidders' evaluations often display a degree of fuzziness. The uncertainty in bid evaluations leads to the use of fuzzy numbers for modeling.

Bids are denoted as $b = (b_1, b_2, \dots)$, where each b_i represents an individual bid. The allocation rule $x(b)$ is a feasible function that maps bids to outcomes in R^n . The payment function $p(b)$, similarly, maps bids to payment values within R^n .

Theorem 1: In a single parameter system, there is a payment rule p for an allocation rule x to fulfill DSIC if and only if allocation rule x is monotone.

Proof:

For all $0 \leq y < Z(r)$, and $z \in Z(r)$, the DSIC condition requires that:

1. When comparing a precise number y with a fuzzy number $Z(r)$, we utilize the α -cut method. Specifically, for any $\alpha \in [0,1]$, the α -cut of $Z(r)$, denoted as $[Z_\alpha^-, Z_\alpha^+]$, must satisfy $y \leq Z_\alpha^-$ for y to be less than $Z(r)$. This ensures a valid comparison.

2. The DSIC condition implies that the utility for any bidder should be maximized by truthfully bidding, hence for any $z > y$:

lowering their bids, it must hold that:

ensuring the DSIC condition.

Combining these two inequalities leads to the conclusion that the allocation rule x must be monotone, which ensures that the mechanism is DSIC.

For any $\alpha \in [0,1]$, the α -cut of the fuzzy number $Z(r)$ is

$$[x(z)Z_\alpha^-, x(z)Z_\alpha^+] - [p(z), p(z)] \geq [x(y)Z_\alpha^-, x(y)Z_\alpha^+] - [p(y), p(y)]$$

This inequality implies two separate inequalities (since $p(z)$ and $p(y)$ are precise numbers):

$$x(z)Z_\alpha^- - p(z) \geq x(y)Z_\alpha^- - p(y) \quad (a)$$

Upper bound:

$$x(z)Z_\alpha^+ - p(z) \geq x(y)Z_\alpha^+ - p(y) \quad (b)$$

Subtract Equation (2) from (a):

$$[x(z)Z_\alpha^- - p(z)] - [y \cdot x(z) - p(z)] \geq [x(y)Z_\alpha^- - p(y)] - [y \cdot x(y) - p(y)]$$

Simplify:

$$x(z)(Z_\alpha^- - y) \geq x(y)(Z_\alpha^- - y) \quad (c)$$

From Equation (b):

$$x(z)(Z_\alpha^+ - y) \geq x(y)(Z_\alpha^+ - y) \quad (d)$$

Since $y \leq Z_\alpha^-$ for all α , we have:

$$Z_\alpha^- - y \geq 0$$

Similarly, because $Z_\alpha^- \leq Z_\alpha^+$, we also have:

$$Z_\alpha^+ - y \geq 0$$

From inequalities (c) and (d), and knowing that $Z_\alpha^- - y \geq 0$ and $Z_\alpha^+ - y \geq 0$, we can deduce:

From (c):

$$(x(z) - x(y))(Z_\alpha^- - y) \geq 0$$

From (d):

$$(x(z) - x(y))(Z_\alpha^+ - y) \geq 0$$

Since $Z_\alpha^- - y \geq 0$ and $Z_\alpha^+ - y \geq 0$, the inequalities imply:

$$x(z) - x(y) \geq 0$$

denoted by $[Z_\alpha^-, Z_\alpha^+]$. To compare a precise number y with $Z(r)$, we ensure that $y \leq Z_\alpha^-$ for all α . Using the α -cut of $Z(r)$, we have:

Lower bound:

Similarly, subtract Equation (2) from (b):

This means that:

$$x(y) \leq x(z) \text{ for all } y < Z(r) \text{ and } z \in Z(r)$$

Thus, the allocation rule x is monotonic.

Now, let's express the difference $[Z(r) \cdot x(z) - Z(r) \cdot x(y)]$ using fuzzy arithmetic.

Compute the difference:

$$\begin{aligned} [Z(r) \cdot x(z)] - [Z(r) \cdot x(y)] &= [x(z)Z_\alpha^-, x(z)Z_\alpha^+] - [x(y)Z_\alpha^-, x(y)Z_\alpha^+] \\ &= [x(z)Z_\alpha^- - x(y)Z_\alpha^+, x(z)Z_\alpha^+ - x(y)Z_\alpha^-] \end{aligned}$$

Since $x(z) \geq x(y)$ and $Z_\alpha^- \leq Z_\alpha^+$, the interval simplifies to:

$$[(x(z) - x(y))Z_\alpha^-, (x(z) - x(y))Z_\alpha^+]$$

Thus, we have

$$\begin{aligned} [z(r), \bar{z}(r)] \cdot x - [\underline{z}(r), \bar{z}(r)] \cdot x &= [x \cdot \underline{z}(r), x \cdot \bar{z}(r)] + [-x \cdot \bar{z}(r), -x \cdot \underline{z}(r)] \end{aligned} \quad (3)$$

Since $y < Z(r)$ and (3) holds, we conclude that:

$$x(y) - x(z) \leq 0 \text{ for all } 0 \leq y < Z(r)$$

Therefore, when (x, p) is DSIC, x must be monotone.

Conversely, if we assume $x(y) - x(z) \leq 0$ for all $y < Z(r)$, then the conditions (1) and (2) are satisfied, ensuring that (x, p) is DSIC. This completes the proof of the first part of Myerson's Lemma.

Theorem 2: An explicit formula can express the payment rule p .

If x is a piecewise constant function, let $y \rightarrow z_1$ where

$$[x(z_1)Z_\alpha^+, x(z_1)Z_\alpha^-] - [p(z_1), p(z_1)] \geq [x(y)Z_\alpha^+, x(y)Z_\alpha^-] - [p(y), p(y)]$$

This indicates that for any level of α , the utility of truth-

$x(z_1) > x(y)$, and let $z = z_1$. Since x is piecewise constant and monotonic, it remains constant over intervals and jumps at certain points. At $z = z_1$, x jumps from $x(y)$ to $x(z_1)$.

For the fuzzy number $Z(r)$, its α -cut is defined as:

$$Z_\alpha = [Z_\alpha^-, Z_\alpha^+]$$

where Z_α^- and Z_α^+ are the lower and upper bounds of the α -level set, respectively.

Knowing $x(z_1) > x(y)$, we must consider the impact of $x(z_1)$ and $x(y)$ on $Z(r)$. Rewrite inequality into α -cut Formulated Inequality:

telling at bid y should be no less than that for any bid z_1 .

We dissect the inequality for each α level:

Lower Bound Handling:

$$x(z_1)Z_\alpha^- - p(z_1) \geq x(y)Z_\alpha^- - p(y)$$

Simplified to:

$$(x(z_1) - x(y))Z_\alpha^- \geq p(z_1) - p(y)$$

Upper Bound Handling:

$$x(z_1)Z_\alpha^+ - p(z_1) \geq x(y)Z_\alpha^+ - p(y)$$

Simplified to:

$$(x(z_1) - x(y))Z_\alpha^+ \geq p(z_1) - p(y)$$

By combining these two simplified inequalities, we derive:

$$(x(z_1) - x(y))Z_\alpha^- \leq p(z_1) - p(y) \leq (x(z_1) - x(y))Z_\alpha^+$$

Dividing by $x(z_1) - x(y)$ (Assuming Non-Zero)

$$Z_\alpha^- \leq \frac{p(z_1) - p(y)}{x(z_1) - x(y)} \leq Z_\alpha^+$$

This indicates that the derived inequality must satisfy the bounds for all possible α -cuts.

And we get:

$$z_1 \leq \frac{p(z_1) - p(y)}{x(z_1) - x(y)} \leq Z(r), \quad \frac{p(z_1) - p(y)}{x(z_1) - x(y)} \in Z(r)$$

Thus, the change in p at z_1 is proportional to $x(z_1)$ multiplied by $Z(r)$. The specific formula for p_i can be expressed as:

$$p_i(b_i, b_{-i}) = \sum_{j=1}^l Z_j(r) \cdot \text{jump in } x_i(b_{-i}) \text{ in the range } [0, b_i] \text{ at } z$$

If x is continuously differentiable, the above result becomes:

$$\frac{dp(z)}{dz} = Z(r) \cdot \frac{dx(z)}{dz}$$

Therefore, the explicit formula becomes:

$$p_i(b_i, b_{-i}) = \int_0^{b_i} \left(Z(r) \cdot \frac{dx_i(z, b_{-i})}{dz} \right) dz$$

Theorem 3: There is a unique payment rule p such that (x, p) is DSIC if we assume $p(b) = 0$.

Proof:

From the first part of the lemma, we already know that (3) ensures DSIC. If a different mechanism (x_1, p_1) does not satisfy the condition from the third part of Myerson's Lemma, it would violate (3). Therefore, the payment rule p must be unique

4. Example: Designing a Simple DSIC Mechanism in CTR

Background:

In internet advertising, click-through rate (CTR) is a critical indicator of an advertisement's effectiveness. Advertisers are often willing to pay more for ads that are expected to generate higher CTRs.

Suppose you manage an online advertising platform that employs a generalized second-price auction mechanism to sell advertising space. The valuation of each advertising space is based on the expected CTR. However, due to market uncertainties, advertisers' estimates of the CTR are often vague, leading them to use fuzzy numbers to represent these estimates.

For example, advertiser A might estimate the CTR of a particular ad slot as a triangular fuzzy number (0.02, 0.03, 0.04). This means that advertiser A believes the most likely CTR is 0.03, but it could range between 0.02 and 0.04.

Scenario:

Consider three advertisers, A, B, and C, who estimate the CTR of an ad slot using the following triangular fuzzy numbers:

- Advertiser A: (0.02, 0.03, 0.04)
- Advertiser B: (0.03, 0.04, 0.05)
- Advertiser C: (0.01, 0.02, 0.03)

Each advertiser submits their fuzzy bids to the platform,

based on their estimations of the ad slot's CTR. Furthermore, the platform has pre-determined the expected CTR values for three available ad slots, which are 1, 0.5, and 0.2, respectively.

Now, another advertiser, D, enters the auction.

Mechanism Rules (based on DSIC principles):

1. Allocation Rule: The n -th advertising space is allocated to the advertiser with the n -th highest bid. In other words, the highest bidder gets the first slot, the second-highest bidder gets the second slot, and so on.
2. Payment Rule: The advertiser who wins the n -th advertising space will pay the $(n + 1)$ -th highest bid. This follows the generalized second-price auction logic, ensuring that advertisers are incentivized to bid truthfully.

5. Conclusion

This paper discusses the core issue of Myerson's lemma under fuzzy valuation in depth. In actual auction and trading scenarios, participants' valuation of goods or services is often not an accurate value but with a certain ambiguity. This ambiguity brings new challenges to mechanism design.

This paper proves and applies Myerson's theory to a fuzzy valuation environment, demonstrating that incentive compatibility mechanisms can still be designed under certain conditions.

For example, in the context of generalized two-price auctions for Internet click-through rates, we show how to achieve the effectiveness and fairness of auctions through reasonable allocation rules and payment rules to ensure that advertisers honestly report their fuzzy valuations.

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