

Using Geometric Group Theory to Find Free Generators of Subgroups of Free Groups

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Abstract: This paper explores how algebraic topology can be used in group theory. More precisely, it describes how the operations known as folding and collapsing on colored directed graphs can be used to find a set of free generators of subgroups of free groups. The technique is illustrated with several examples.

Keywords: Geometric Group Theory, Free Group, Subgroup, Free Generator, Graph Folding, Graph Collapsing

1. INTRODUCTION

A common theme of pure mathematics is to define abstract classes of objects motivated by concrete examples, then study those abstract classes to uncover their general properties, to finally make conclusions on the particular examples of interest. A powerful strategy is to use together apparently different areas of mathematics to obtain results that would otherwise be difficult to obtain.

In this article this article studies free groups. The problems this paper seeks to solve are:

1. Show that any subgroup of a free group is also free
2. Given a set of generators of a subgroup of a free group, obtain a set of free generators of that subgroup

The precise definitions of the terms mentioned above: groups, subgroups, free groups, generators and free generators are given later in the article. The reader is also referred to the books [1, 2] for more details. To accomplish these tasks, elements of graph theory and algebraic topology are used.

The study of this article belongs to the area of mathematics known as geometric group theory [3]. Groups are algebraic structures and thus, are considered part of abstract algebra. The term geometric in geometric group theory refers to the fact that this area uses algebraic topology [4] in graphs [5], which is the study of closed curves on graphs. All these concepts are explained at length in this article.

This article is an example of the power of abstraction of objects and the power of connecting different fields of mathematics. The problem addressed in this article is just one of many interesting research questions in the very active field of geometric group theory.

2. BACKGROUND

2.1. Group and Subgroup

An algebraic structure with one binary operation $\langle G, * \rangle$ is called a group if,

1. For all a, b in G , then $a*b$ is in G .
2. For a, b, c are in G , then $a*(b*c) = (a*b)*c$
3. There exists an identity element e in G such that $a*e = e*a = a$ for all a in G .
4. For every element in G , there exists a^{-1} with $a*a^{-1} = a^{-1}*a = e$.

Suppose G is a group. A subset H of G is a subgroup of G if H is a group with the same group operation as G .

Equivalently, a subset H of a group G is a subgroup if and only if the following are satisfied:

1. e belongs to H
2. if a belongs to H , so does a^{-1}
3. if a and b belong to H , so does $a*b$

2.2. Free Group

Let $S = \{a, b\}$ be a set with two elements. The words on a, b, a^{-1} and b^{-1} are the sequences on these elements, for example $aaa^{-1}b^{-1}ba^{-1}$ is a word. A word is irreducible if neither a and a^{-1} nor b and b^{-1} appear in the word consecutively. As the sections in red illustrate, the word $aaa^{-1}b^{-1}b^{-1}ba^{-1}$ is not irreducible.

A word that is reducible if adjacent inverse pairs such that a and a^{-1} or b and b^{-1} each time. Every word reduces uniquely to an irreducible word, regardless of the order of cancellation. For example, if aa^{-1} is cancelled in the word $aaa^{-1}b^{-1}b^{-1}ba^{-1}$ the result is $ab^{-1}b^{-1}ba^{-1}$.

A reducible word can be reduced to an irreducible word by canceling a and a^{-1} each time they are next to each other in the word, and canceling b and b^{-1} each time they appear consecutively in the word. While cancelling these adjacent words, they simplify into identity e . For example, $aaa^{-1}b^{-1}b^{-1}ba^{-1}$ will simplify to aba^{-1} .

Given a set, a free group can be built from that set. As an example, assume the set has two elements $\{a, b\}$. Let G be the free group from that set. The elements of G are the irreducible words on a, b, a^{-1} and b^{-1} . The irreducible words are the sequences that do not contain adjacent pairs like a, b, a^{-1}, b^{-1} .

A free group G is built from a set $\{a, b, \dots\}$ of generators i.e. $\{a, b, \dots\}$, No simplification is allowed except cancellation of adjacent inverses.

Let $F = \{a_1, a_2, \dots, a_n\}$ is a finite set. The free group on F denoted as F_n consists of all reduced forms using the letters a_i for i represent different generators in the set, and their inverse a_i^{-1} , with multiplication defined as concatenation reduction, meaning that pairs of adjacent inverses (e.g. cancelling $a_i a_i^{-1}$ or $a_i^{-1} a_i$ into identity).

For example, let $F = \langle a, b \rangle$ is a free group. Then $ab^{-1}a$ is reduced since it can not be simplified. $aba^{-1}a$ can be reduced to ab .

2.3. Operator and Free Generating Set

Let G be a group. A set $S = \{g_1, g_2, \dots, g_n\}$ is a generating set if every element of G can be written as a finite product of

g_i and their inverse. A free generating set means that only relations among its elements are trivial, which means it can cancel with inverse. In other words, no non-trivial relations exist among the generators.

For example, $F = \langle a \rangle$, then a is a free generator. But $\{a, a^2\}$ is not a free generating set, since $a^2 = a * a$, so it is reducible.

2.4. Directed Graph and Path

A directed graph Γ is a pair of sets, $\Gamma = (V, E)$, where V is a set of vertices and E is the set of edges. Each edge can have either one or two endpoints (Figure 1), that is vertices. If edge e has two endpoints, one is its initial vertex and the other is terminal vertex. If edge e has one endpoint, then this vertex is both its initial vertex and terminal vertex and such edge is called a loop (Figure 2). All edges have direction. Edges that are not loops have a direction from the initial vertex towards the terminal vertex. Loops are assigned a direction.

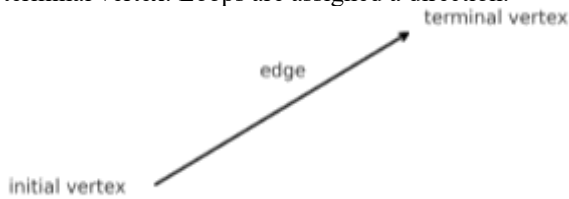


Figure 1. Edge.

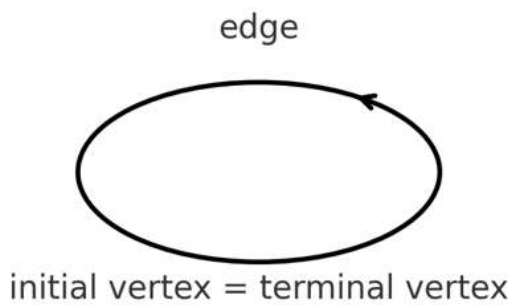


Figure 2. Loop.

Each edge e in a directed graph Γ has a reverse, denoted by e^{-1} which shares same endpoints as edge e but has opposite direction. For edges that are not loops, the terminal vertex of e^{-1} is the initial vertex of e and the initial vertex of e^{-1} is the terminal vertex of e (Figure 3). Loops have a reverse that is just the opposite direction with the same initial vertex that is the terminal vertex (Figure 4). For any edge e , the reverse of e^{-1} to be e and the notation $(e^{-1})^{-1} = e$ is used.

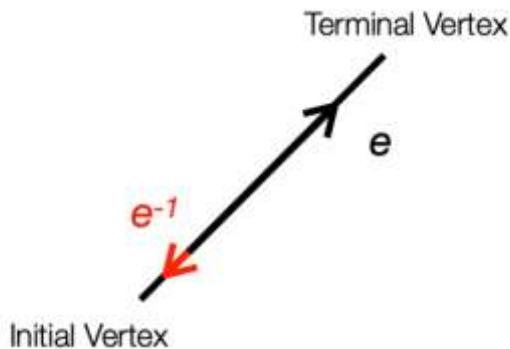
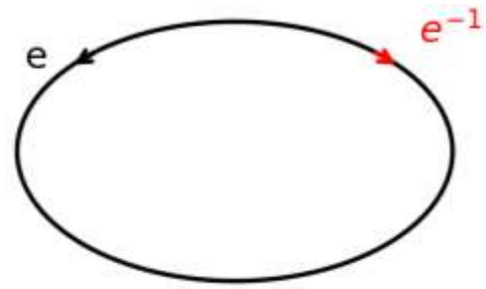


Figure 3. Reverse Edge.



initial vertex e = terminal vertex e^{-1}

Figure 4. Reverse Loop.

A path is a finite sequence of edge $e_0 e_1 \dots e_k$ such that each e_i ($0 < i \leq k$) is an edge or the reverse of an edge with the terminal vertex of e_{i-1} being the initial vertex of e_i . A path is closed if its terminal vertex is its initial vertex.

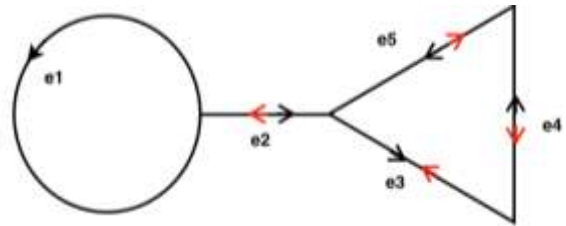


Figure 5. Path Example.

In the Figure 5 above, $e_4 e_5 e_2^{-1}$ in red is an example of a path.

2.5. Close path, Tight Path

A closed path is a path whose terminal vertex is its initial vertex. For a closed path, the terminal vertex that is its initial vertex is called the base vertex. In the Figure 5, if $e_2 e_3 e_4 e_5 e_2^{-1}$ is chosen, the path in red, is a closed path with its base vertex v .

A path is tight if there is no reverse of an edge immediately following that edge (i.e. no edge is directly followed by its reverse). Every path can be tightened by removing backtracking sub paths. In the Figure 5, it can be seen that $e_2 e_3 e_4 e_5 e_2^{-1}$ is a tight path but $e_1 e_2 e_2^{-1}$ is not a tight path since the ' $e_2 e_2^{-1}$ ' makes it not tight (Figure 6).

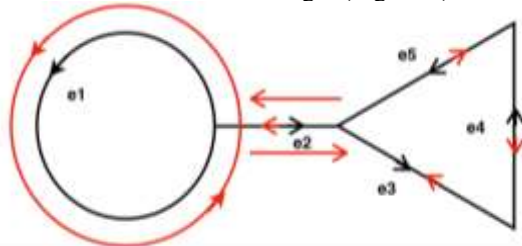


Figure 6. $e_1 e_2 e_2^{-1}$ is not a tight path.

The process of tightening is removing all such ' $e_2 e_2^{-1}$ ' such that a new path e_1 that is tight is obtained (Figure 7).

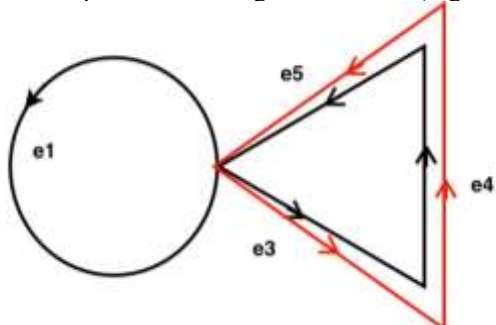


Figure 7. $e_2 e_2^{-1}$ is a tight path.

2.6. Fundamental Group of Graph

Let Γ be a directed graph. Let v be a vertex of Γ . $\pi(\Gamma, v)$ is the set of all closed tight paths based at vertex v .

The following multiplication on $\pi(\Gamma, v)$ is defined: Let a and b be elements of $\pi(\Gamma, v)$. Then $a * b$ is defined as first concatenating a and b , and then tightening the resulting path. It is known that $a * b$ must be a closed tight path.

In the following Figure 8, there are closed tight path $e_2e_3e_4e_5e_2-1$ and $e_2e_3e_4e_5e_2-1$ all based at vertex v that belongs to $\pi(\Gamma, v)$. Under multiplication, $(e_2e_3e_4e_5e_2-1) * (e_2e_3e_4e_5e_2-1)$ is equivalent to first concatenating: $e_2e_3e_4e_5e_2-1e_2e_3e_4e_5e_2-1$ and then tightening: $e_2e_3e_4e_5e_2e_3e_4e_5e_2-1$. It can be seen that $e_2e_3e_4e_5e_2e_3e_4e_5e_2-1$ is a closed tight path based at vertex v , so $\pi(\Gamma, v)$ is closed under multiplication.

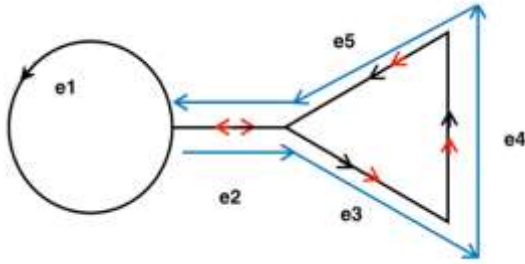


Figure 8. $(e_2e_3e_4e_5e_2-1) * (e_2e_3e_4e_5e_2-1)$ is a closed tight path.

The path with no element (starting in v and immediately ending the path without moving along any edges) is included in $\pi(\Gamma, v)$. With this multiplication, $\pi(\Gamma, v)$ forms group. The identity in the group is the trivial path at v , and the inverses are given by reversing paths. This is called the fundamental group.

2.7. Graph Map and Homomorphism

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be two directed graphs. A graph map $f: \Gamma \rightarrow \Gamma'$ is a function that maps vertices and edges in Γ to vertices in Γ' that satisfies:

$f(\text{initial vertex of } e) = \text{initial vertex of } f(e)$

$f(\text{terminal vertex of } e) = \text{terminal vertex of } f(e)$

Homomorphism

The function $f^*: \pi(\Gamma, v) \rightarrow \pi(\Gamma', f(v))$ is defined as follows: If $a = e_0e_1\dots e_k$ belongs to $\pi(\Gamma, v)$, then $f^*(a)$ is equal to the tight closed path that results from tightening $f(e_0)f(e_1)\dots f(e_k)$.

For example, the Figure 9 below shows graph homomorphism, that shapes change but preserve adjacency.

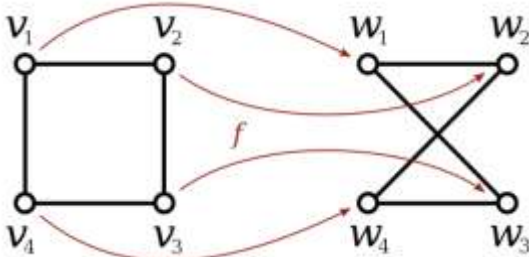


Figure 9. Homomorphism.

A graph map induces a homomorphism:

$f: \pi(\Gamma, v)$ and $\pi(\Gamma', f(v))$

It sends a path to its image under f , followed by tightening. This guarantees that $f^*(ab) = f^*(a)f^*(b)$.

2.8. Colored Directed Graph

Colored directed graphs are directed graphs where each

edge has a color. A colored directed graph is one in which each edge is assigned a color. A colored directed graph map preserves both adjacency and edge colors.

The following Figure 10 shows two colored graphs Γ and Γ' .

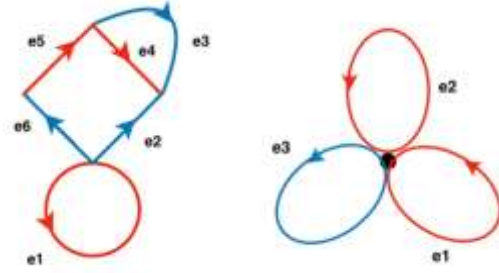


Figure 10. Mapping graph from $\Gamma \rightarrow \Gamma'$.

There can be the following two-colored graph maps f_1 (table 1 on the left) and f_2 (table 1 on the right): $\Gamma \rightarrow \Gamma'$.

Table 1. Definitions of two distinct colored graph maps, f_1 and f_2 , from graph Γ to Γ' .

edge	$f_1(\text{edge})$	edge	$f_2(\text{edge})$
e1	e1'	e1	e2'
e2	e3'	e2	e3'
e3	e3'	e3	e3'
e4	e1'	e4	e1'
e5	e2'	e5	e2'
e6	e3'	e6	e3'

Constructing Graph from A Sequence

By a sequence of $\{r, b, r-1, b-1\}$, is meant a sequence $a = a_1a_2\dots a_k$ such that each a_i ($1 \leq i \leq k$) is either $r, b, r-1, b-1$. Given a sequence a on $\{r, b, r-1, b-1\}$, a directed colored graph $\Gamma(a)$ can be constructed as follows.

The process starts with a base vertex v , then go around in counterclockwise direction adding one edge per element in the sequence. If the element is r or $r-1$, then the color of the edge is red. Otherwise, it is blue. If the element is r or b , the direction of the edge is counterclockwise. Otherwise, it is clockwise.

For example, if there are sequences $\langle a_3b\dots \rangle$ from a common vertex, letting a be red and b be blue, each sequence can be constructed. The graph constructed looks as follows (Figure 11):

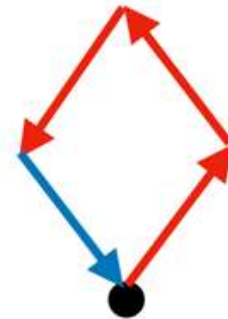


Figure 11. a_3b .

2.9. Folding

If two edges of the same color share the same initial vertex (or terminal vertex), then it can be folded into one edge. Folding simplifies the graph while preserving the fundamental group. Repeated folding produces a fully folded graph, where no further identification is possible.

For instance, the following figure shows a series of folding. In Figure 12, from step 1, the two red edges coming from the same initial vertex labelled in yellow are taken and folded to reach step 2. From step 2 it can be seen that there are two red edges with the same terminal vertex labelled in yellow; they can be folded to get step 3. On a given color directed graph constructed from some sequence, folding can be done to the graph until no edges of the same color have the same initial or terminal vertex.

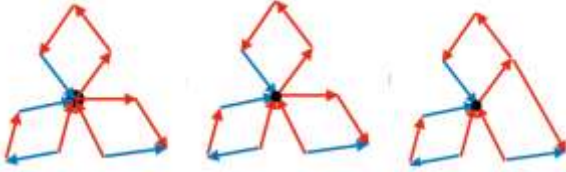


Figure 12. Folding (Step 1-3).

2.10. Collapsing

Collapsing a non-loop edge merges its two endpoints into a single vertex and removes the edge, re-attaching all other edges accordingly.

Here is the mathematical expression:

Let $\Gamma = (V, E)$ be a directed graph. Let e be an edge that is not a loop. The graph after collapsing is denoted as $\Gamma_{\downarrow e} = (V_{\downarrow e}, E_{\downarrow e})$ and defined as follows:

1. Vertices of $\Gamma_{\downarrow e}$ are $V_{\downarrow e} = V - \{\text{endpoints of } e\} \cup \{v_{\downarrow e}\}$, where $v_{\downarrow e}$ is a new vertex

2. The edges of $\Gamma_{\downarrow e}$ are $E_{\downarrow e} = E - \{e\}$

3. Let e' belongs to $E_{\downarrow e}$:

3.1. If the initial vertex of e' has an edge of Γ that is not an endpoint of e , then the initial vertex of e' as an edge of $\Gamma_{\downarrow e}$ is the same as it is in graph Γ

3.2. If the terminal vertex of e' has an edge of Γ that is not an endpoint of e , then the terminal vertex of e' as an edge of $\Gamma_{\downarrow e}$ is the same as it is in graph Γ

3.3. If the initial vertex of e' as an edge of Γ is an endpoint of e , then the initial vertex of e' as an edge of $\Gamma_{\downarrow e}$ is $v_{\downarrow e}$.

3.4. If the terminal vertex of e' as an edge of Γ is an endpoint of e , then the terminal vertex of e' as an edge of $\Gamma_{\downarrow e}$ is $v_{\downarrow e}$.

3.5 Natural isomorphism

Let $\Gamma = (V, E)$ be a directed graph. Let v be a vertex and let e be an edge that is not a loop. If v is not an endpoint of e , then set $v' = v$. If v is an endpoint of e , set $v' = v_{\downarrow e}$. Let a belongs to $\pi(\Gamma, v)$. By removing all e and e^{-1} that appear in the tight sequence that forms a , a new sequence of edges and reverse edges can be obtained which is called $a_{\downarrow e}$. $a_{\downarrow e}$ is tight and so $a_{\downarrow e}$ belongs to $\pi(\Gamma_{\downarrow e}, v)$. Let $f_{\downarrow e}: \pi(\Gamma, v) \rightarrow \pi(\Gamma_{\downarrow e}, v)$ defined by $f_{\downarrow e}(a) = a_{\downarrow e}$ shows group isomorphism. Therefore, collapsing does not change the fundamental group.

Taking the example left from folding, which will be discussed later in section 4.3, after a series of folding the color directed graph in step 7 that cannot be further folded but can be further collapsed. From step 7, the edge e_4 is collapsed, which brings vertex v_4 to v_3 and obtains the graph in step 8. Then edge e_2 can be collapsed, bringing vertex v_1 to v_5 . Steps of collapsing can be repeated to a point where no collapsing can be further done. That is when there are no other vertices except the common vertex, giving us a n -rose graph.

2.11. N-Rose Graph

A rose graph with n petals is a graph consisting of a single vertex with n loops. By folding and collapsing, any subgroup

can be transformed into a rose. The loops of the resulting rose correspond to a free generating set for the group. The belloved Figure 13 shows examples of n -rose graph.

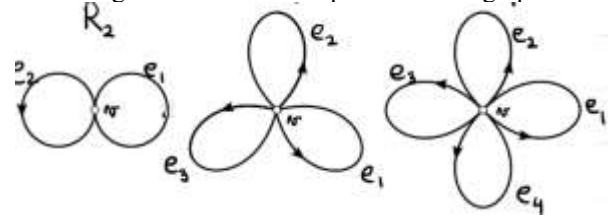


Figure 13. N-rose Graph.

For example, given a set sequence of $\{r, b, r^{-1}, b^{-1}\}$, a directed colored graph can be constructed. First, the graph of each sequence is constructed, then those graphs are connected by a sharing base vertex without sharing any other vertex or edge.

The graph of a finite set S is a proper subset of $\pi(R_2, v)$ since each element of $\pi(R_2, v)$ is a sequence of $\{r, b, r^{-1}, b^{-1}\}$ and the graph of the finite set S is the graph of the set of these sequences (Figure 14).

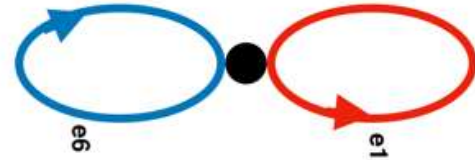


Figure 14. N-rose.

In this Figure 14, this is a n -rose graph that was obtained after a series of folding and collapsing. There exists only a single common vertex of v_3 and loops as edges of e_1 and e_6 .

3. ALGORITHM FOR COLORED GRAPH MAPPING TO FIND A SET OF FREE GENERATORS OF SUBGROUPS OF FREE GROUPS

3.1. Motivation: why translate groups into graphs?

To understand subgroups in free groups, it is useful to translate algebraic elements into geometric representations. Consider a free group on two generators, $F_2 = \langle a, b \rangle$, whose elements are words in a, b and their inverses, such as $a^{-1} = ab^{-1}$ or $a^2 = b^2a^{-1}$.

Suppose there is a subgroup $H \subset F_2$ is generated by $H = \langle a_1, a_2 \rangle$, then every element of H can be formed by finite combinations of a_1, a_2 , and their inverses. A natural question is what is the precise structure of the subgroup H , and can a free generating set for H be found?

One approach is to treat words as paths on a graph. This method creates a bridge between abstract algebra and geometry. By visualizing subgroups as paths and loops within graphs, geometric techniques—such as graph folding—can be used to clarify subgroup structure and visually identify free generators.

3.2. Use graphs that represent free groups

A useful tool for representing free groups is the rose graph: a graph with a single vertex and n loops, each loop corresponding to one generator.

For example, in $F_2 = \langle a, b \rangle$, let a subgroup $H = \langle a_1, a_2, a_3 \rangle$, where $a_1 = a_2b$, $a_2 = a_2a^{-1}$, $a_3 = \dots$, and so on. Each generator corresponds to a specific sequence of the letters a and b which colors correspond to red and blue, with inverse indicated by reversing orientation.

In this setting, a loop at the basepoint represents a word in the group. By restricting to tight loops, it is ensured that the paths correspond to reduced words. This guarantees that the collection of such loops is isomorphic to the free group F_2 .

Thus, the fundamental group of the rose graph is isomorphic to the free group F_n . The graph serves as a geometric model of the group, allowing algebraic properties to be studied through visual and combinatorial means.

3.3. Constructing a graph from sub-group generators

Considering the subgroup H is a proper subset of F_2 , generated by $a_1 = ab^{-1}a$ and $a_2 = b_2a^{-1}$.

Each generator is represented as a path:

- $ab^{-1}a$: red forward edge, blue backward edge, red forward edge
- b_2a^{-1} : two blue edges. red backward edge.

These paths are joined at a common basepoint to form the subgroup graph. Its fundamental group is isomorphic to H . In simplification steps, folding and collapsing will simplify this graph and reveal a free generating set.

3.4. Colored Graph Map

What is colored graph map

A colored graph map $f: \Gamma \rightarrow R$ formalizes the connection between subgroup graph Γ and a rose graph R . It satisfies:

1. Vertex preservation: All vertices in Γ are mapped to the single vertex in R .
2. Edge direction preservation: The direction of each edge in Γ is maintained in R .
3. Edge color preservation: The color (i.e., generator label) of each edge is preserved in the image.

Thus, a red forward edge maps to a red loop traversal, while a blue backward edge maps to a reversed traversal of the blue loop.

This map induces a homomorphism on fundamental groups: $f_*: \pi_1(\Gamma, v) \rightarrow \pi_1(R, v) \cong F_n$

Whose image is the subgroup H , the subgroup graph encodes H geometrically and folding operations respect the algebraic structure (Figure 15).

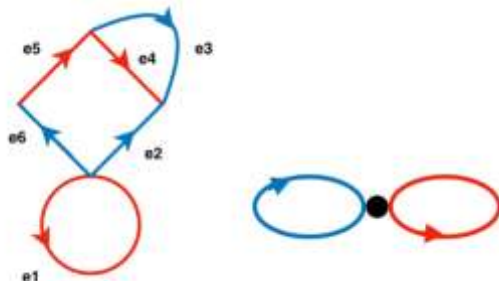


Figure 15. Mapping from Γ to Γ' .

3.5. Folding and Collapsing

Folding simplifies a subgroup by merging edges of the same color and orientation that share a vertex, this eliminates redundancy while preserving the subgroup. Folding continues until no further identification are possible, in the resulting folded graph:

- Each loop at the basepoint corresponds to a generator of

the subgroup.

- These loops are freely independent
- Together, the loops form a free basis for the subgroup.

Once folding is complete, collapsing is applied: a simplification that shortens longer back-and-forth paths into direct connections, further reducing the graph without changing its fundamental group. Repeated collapsing proceeds until the graph is transformed into an n -rose—a single vertex with n loops. These loops form a free generating set for the subgroup H .

4. CONSTRUCTION STRATEGY: HOW TO FIND FREE GENERATORS OF A SUBGROUP

4.1. Building the Colored Directed Graph

To begin, a set of subgroup generators is taken, considering $F_2 = \langle a, b \rangle$, for $H = \langle a_3b, a_2b^{-1}a, a^{-1}bab \rangle$, and a directed graph is constructed where each generator is represented as a path:

The color directed graph can be built from a common vertex:

- The letter a corresponds to a red edge, and b to a blue edge.
- Inverses are represented by reversing the direction of the corresponding edge.

Each word is drawn as a sequence of colored edges starting from a common vertex. This forms the initial subgroup graph.

Here is the construction of the original graph: (step 1, figure 12)

4.2. Folding

Next, edges in the graph that meet the following criteria are identified and folded:

- Share the same starting vertex or ending vertex.
- Have the same color.

This process simplifies the graph while preserving the subgroup it represents. Folding is repeated iteratively—merging overlapping or redundant paths—until no further folds are possible.

The resulting folded graph is structurally simpler but retains the same fundamental group. From this graph, one can directly read off a free generating set for the subgroup. These generators correspond to the distinct loops in the simplified graph, which now serves as a minimal and visual representation of the subgroup H .

Recognizing edges with the same color as the common initial vertex or terminal vertex labelled in yellow, the following series of folding from step 1 to step 6 can be done (Figure 16).

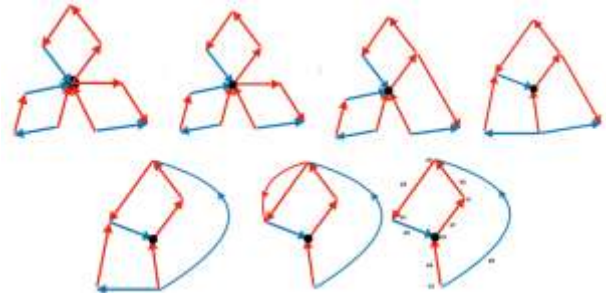


Figure 16. Folding step (Row 1 from left to right: Step 1 to 4; Row 2 from left to right: Step 5 to 7).

A color directed graph that cannot be further folded is obtained shown in Figure 16 Step 7.

4.3. Collapsing into a Rose

After folding, some edges may still form paths rather than loops. Collapsing identifies all vertices along such paths with a basepoint. Collapsing continues until every edge is a loop at the basepoint, yielding a rose graph.

Once the graph becomes a rose, the end of the construction has been reached. Each loop in the rose corresponds to a closed path, hence, it constructs a sequence in the free group. And the sequences form a free generating set for the subgroup.

In the example followed by the end of folding, the graph will then be collapsed as follows: e_4, e_2, e_3, e_5 are collapsed in order.

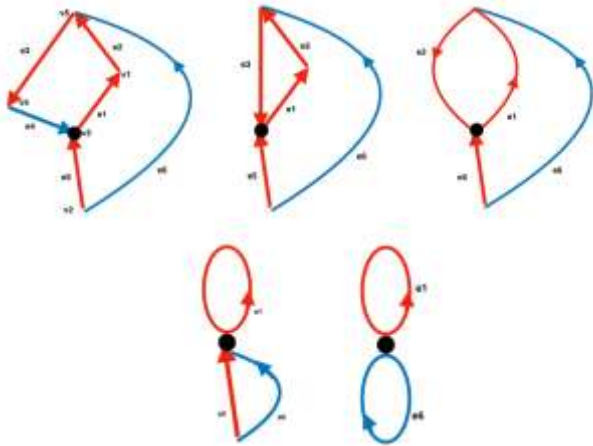


Figure 17. Collapsing into n-rose graph.

The graph obtained at step 11 is a n-rose graph, with the two loops e_1, e_6 the free generating set of free group H (Figure 17).

In step 11, e_1 and e_6 were obtained as final loops associated with 2-rose graphs. It is necessary to find the tight closed path in Γ_0' based at v_3 that maps e_1 and e_6 as edges of Γ_1' . This will give us a free generating set of $\pi(\Gamma', v_3)$. As shown in Figure 17, in Γ_1' e_6 and e_1 are left. In Γ_0' , e_1 corresponds to $e_1e_2e_3e_4$, and e_6 corresponds to $e_5-1e_6e_3e_4$. Listed in the

following Table 2.

Table 2. Correspondence between the free generators in the final n-rose graph (Γ_1') and their original path representations in the pre-collapsing graph (Γ_0').

In Γ_1'	e_1	e_6
In Γ_0'	$e_1e_2e_3e_4$	$e_5-1e_6e_3e_4$

5. CONCLUSION

This paper has clearly illustrated a way to use geometric group theory, specifically graph folding and collapse, to answer a basic question of abstract algebra: finding free generators for a subgroup of a free group. By interpreting the algebraic notion of group generators as paths on a directed vertex colored graph, the problem is turned into a geometric problem that can be easily implemented algorithmically and can use geometric intuitions, too. The process of graph folding and collapsing edges takes the complicated starting graph to its simple form of an n-rose graph. The loops of this final graph represent the elements of the free generating set for the subgroup, providing an attractive and visual approach to the structural analysis of free groups.

6. REFERENCES

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