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Group Theory and Patterns Generation in High Order Rubik Cubes

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Abstract: This paper explores the relationship between group theory and Rubik's Cube operations and explains how mathematical structures can be used to generate patterns on higher-order cubes. The operations of a Rubik's Cube can be modeled by a finite permutation group, where each move corresponds to a permutation of edge and corner blocks. By analyzing the locations and orientations of these blocks, the paper explains why repeated operations eventually return the cube to its original state. The total number of valid configurations of a 3×3×3 Rubik's Cube is derived using permutation constraints on edge and corner positions and orientations. Examples such as the operations U, RRUU, and FR are used to illustrate how permutation cycles determine the order of operations and the resulting cube patterns. The study further extends these ideas to higher-order cubes and demonstrates methods for generating visual patterns on a 7×7×7 cube using structured sequences of layer rotations.

Keywords: Group theory; Rubik's Cube; Permutation group; Cube operations; Pattern generation.

1. Introduction

The Rubik cubes are a type of game that players intend to rearrange disordered blocks back into their original order [1]. It is invented in the mid-1970s by Erno˝ Rubik as a model for spatial reasoning and problem solving. During the process of attempting to solve the Rubik cubes, it is found that some operations result in a regular swap of the locations of some specific blocks (this will be explained in detail later). More surprisingly, repeating a specific moving pattern will always turn the cube back to the original situation within infinite number of moves. This is because the cube's operation set can be modeled by a finite group, where every move sequence is an element of finite order, and so repeatedly applying any fixed algorithm will eventually return the cube back into its original situation. In fact, group theory, especially permutation groups, is closely related to Rubik cubes. This essay will focus on the reason behind these phenomenon and develop a way of generating patterns on high order Rubik cubes.

2. Group Theory

A group is a nonempty set G equipped with a binary operation ∗ satisfying the following axioms (Cornwell 1997)[2]:

1. Closure: If a, b ∈ G, then a ∗ b ∈ G;
2. Associativity: (a ∗ b) ∗ c ⇔ a ∗ (b ∗ c);
3. Identity element: ∃e ∈ G, such that for any a ∈ G, a ∗ e = e ∗ a = a;
4. Inverse: ∀a ∈ G, ∃b ∈ G, such that a ∗ b = b ∗ a = e.

Some well-known examples of groups are integers under addition, real/complex numbers except for 0 under multiplication. These groups are infinite groups and abelian groups, as they have infinite number of elements and are commutative (a ∗ b = b ∗ a). An example of a finite commutative group is the complete residue class mod n under addition.

Apart from numbers, other elements and operation can also form groups. For the equilateral triangle shown below, define

1. e: Leave it unchanged;
2. r1: Rotated it 120◦ clockwise;
3. r2: Rotated it 240◦ clockwise;
4. s1: Reflect it about the perpendicular bisector of BC;
5. s2: Reflect it about the perpendicular bisector of AC;
6. s3: Reflect it about the perpendicular bisector of AB;

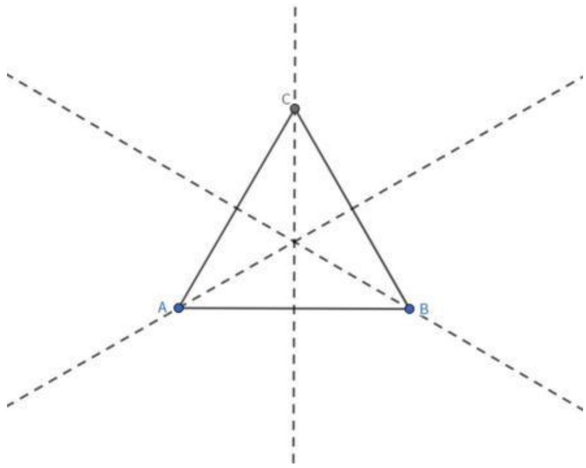

Figure 1. Symmetry operations of an equilateral triangle

Obviously e is the identity element, and the composition of the defined operations follow asso- ciativity. r1 and r2 are inverse of each other and s1, s2 and s3 are inverse of themselves.

*	e	r1	r2	s1	s2	s3
e	e	r1	r2	s1	s2	s3
r1	r1	r2	e	s3	s1	s2
r2	r2	e	r1	s2	s3	s1
s1	s1	s2	s3	e	r1	r2
s2	s2	s3	s1	r2	e	r1
s3	s3	s1	s2	r1	r2	e

The most important concept of group theory in Rubik cubes is the permutation group. Per- mutation is to rearrange a series of sequential elements. For example, consider the numbers {1, 2, 3, 4, 5, 6}.

{1, 2, 3, 4, 5, 6} → {4, 2, 1, 3, 6, 5}

As shown, 1 moves to 3, 3 moves to 4 and 4 moves to 1, while 5 moves to 6 and 6 moves to 5.

This permutation can be expressed as

$$(2)(1, 3, 4)(5, 6)$$

For the composition of permutations, consider {1, 2, 3} and permutations (1, 2) and (1, 3).

$(1, 2) * (1, 3)$ means 1 moves to 2 while 2 moves to 1 then moves to 3 and 3 moves to 1.

So $(1, 2) * (1, 3) = (1, 2, 3)$. However, $(1, 3) * (1, 2) = (1, 3, 2)$, indicating permuta- tions are not commutative. Additionally, it is not hard to find by induction that every permuta- tion can be expressed as a composition of some second-order permutations, $(x1, x2, \cdots xn) =$

$(x1, x2)(x1, x3) \cdots (x1, xn)$. If a permutation can be expressed as a composition of an even

number of second-order permutations, we call it an even permutation. Otherwise we call it an odd permutation.

3. The Rubik Cube and Some Important Notations

The operations on a Rubik cube can be modeled by a group. First define the six faces of the cube as front(F), back(B), left(L), right(R), up(U) and down(D).

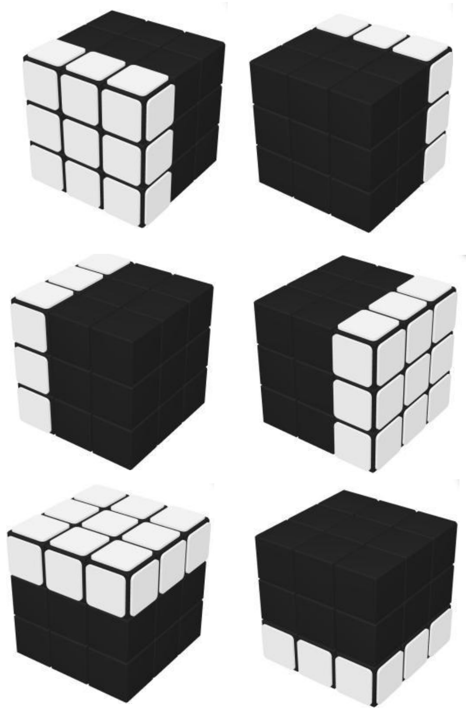

Figure 2. Notation of the six faces of a Rubik's Cube

Each operation means a 90◦ rotation clockwise (when facing that face). Below shows what happens with the operation U.

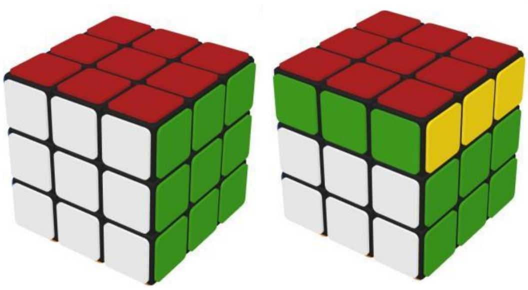

Figure 3. Effect of the operation U on a Rubik's Cube

The Rubik cube is consisted of 6 center blocks, 12 edge blocks, and 8 corner blocks.

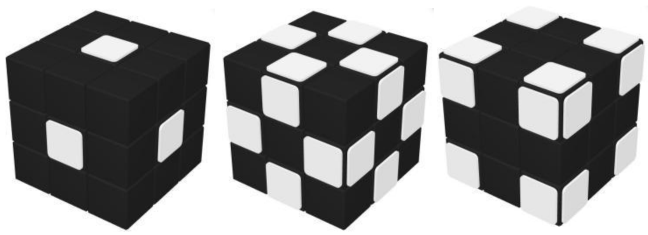

Figure 4. Components of a Rubik's Cube: centers, edges, and corners

Note that any operation will not affect the location or direction of the 6 center cubes, so we only need to consider the edge cubes and the corner cubes. For location, each cube is defined by the faces that include them. As an example, the edge cube in the left figure is uf, and the corner cube in the right figure is ufr.

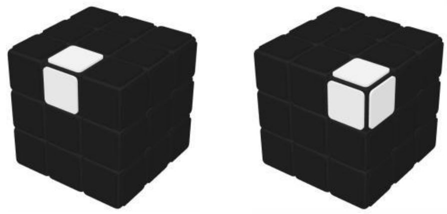

Figure 5. Notation for the positions of edge and corner blocks

For direction, note that the direction of a cube is fixed once a specific face is fixed. Without loss of generality, consider the white or yellow face of each cube, and the color of front and back for the four edge cube that do not have while and yellow color. Let the face that have the chosen color be capital letter. As an example, the edge cube in the left figure is ufR, and the corner cube in the right figure is Ufr.

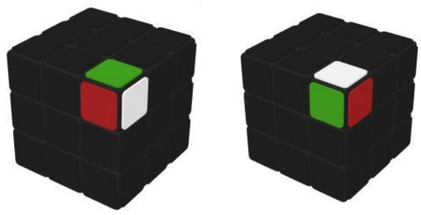

Figure 6. Notation for the orientations of edge and corner block

4. How Group Theory Works in Rubik Cube

To model the Rubik cube with group theory, first we can consider all possible operations as a group. The identity element of this group is doing nothing to the cube. Apparently the composition of any two operations is still an operation and belongs to the group. The inverse of any operation is to undo the operation reversely, which is also an operation and belongs to the group. For convenience, define the inverse of the 6 basic operation U, D, F, B, L and R as U', D', F', B', L' and R'. Note that rotation 270◦ clockwise is equivalent to 90◦ anti-clockwise, so UUU=U'. If the cube is identical after two operations, we consider the two operations the same. For example, UUU=U', F=FFFFF. So how many elements are there is this Rubik cube group, in other words, how many possible situations are there for a 3 × 3 × 3 Rubik cube?

Theoretically, the 12 edge blocks have 12! possible locations, and each edge block have 2 pos- sible directions. Similarly, the 8 corner blocks have 8! possible locations, and each corner block have 3 possible directions. So that is 12! × 212 × 8! × 38 situations in total. However, some of the cases is not possible, such as the two figures shown below.

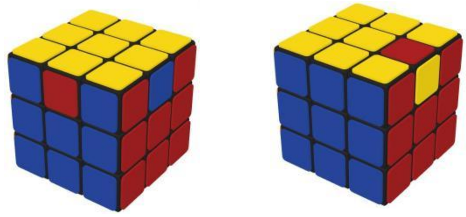

Figure 7. Examples of invalid Rubik's Cube configurations

First we consider the location of the blocks. A basic operation on the cube, for example U, generates the following permutation.

(uf, ul, ub, ur)(ufr, ufl, ubl, ubr)

Note that this is an even permutation. Therefore, any operations, as they are compositions of the six basic operations, results in an even permutation. This indicates that it is possible to swap two pairs of blocks, but not possible to swap the location of a single pair of blocks. Thus, only half of the total situations are valid.

For the direction of the edge blocks, label each block with the direction of the corresponding x,y,z-axis, as shown.

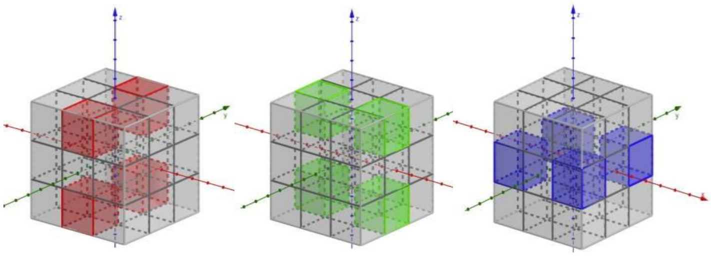

Figure 8. Orientation labeling of edge blocks using coordinate axes

It is not hard to find that each basic operation results in two blocks changing direction, so any composition of operations will result in an even number of blocks changing direction. Thus, half of the total number of situations are invalid as a single block changing direction is not possible. Similarly, each corner block has three possible directions and only one third of the total situations are valid for the corner blocks. Therefore, the total valid number of situations would be

$$\frac{12! \times 2^{12} \times 8! \times 3^{8}}{2 \times 2 \times 3} = 43252003274489856000$$

Now we know that the operations on a Rubik cube can be modeled by a finite group with 43252003274489856000 elements [3]. So why repeating a fixed operation will always turn the cube back to its original order [4]. We start with the most basic case, the basic operation U. We consider the permutation it generates

(uf, ul, ub, ur)(ufr, ufl, ubl, ubr)

Note that this is two independent permutations each with order 4. So after 4 operations, all blocks return to their original space. And it is indeed the case since UUUU is rotation of the up surface $90^\circ \times 4 = 360^\circ$.

Next we study the operation RRUU. The permutation generated by this operation is

(fr, br)(uf, ub)(dr, ul, ur)(dfr, ufl, udr)(ufr, dbr, ubl)

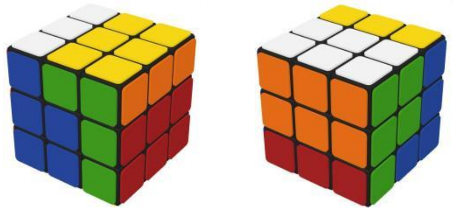

Figure 9. Permutation cycles generated by the operation RRUU

This is a permutation consisted of two second-order permutations and three third-order permu- tations. Since the least common multiple of 2 and 3 is 6, after repeating the operation 6 times, all blocks will be back into their original order. The permutation displayed also illustrates why after repeating the operation 3 times, as shown below, the cube forms a relatively regular pattern, since the blocks in the three third-order permutations are back into their original space.

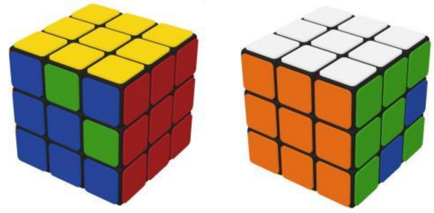

Figure 10. Pattern formed after repeating the operation RRUU three times

Note that both operation U and operation RRUU will not change the direction of the blocks.

Lastly, we study the operation FR. Regardless of direction, the permutation generated is

(uf, ur, br, dr, fr, df, fl)(ufr)(ufl, ubr, dbr, dfr, dfl)

The least common multiple of 7,1 and 5 is 35. However, after repeating the operation 35 times, the cube does not return back into its original situation.

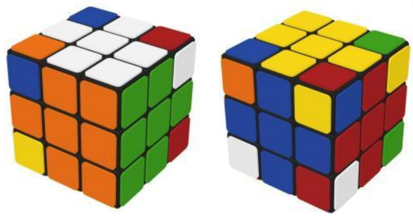

Figure 11. Cube configuration after repeating the operation FR 35 times

As shown, each block is indeed back into their original location, but the 6 corner blocks involved are not in the right direction. Considering direction, the permutation turns into

(Uf, uR, bR, dR, fR, Df, fL)(uF, Ur, Br, Dr, Fr, dF, Fl)
(Ufr, ufR, uFr)
(Ufl, ubR, dbR, dfR, Dfl, ufL, uBr, Dbr, dFr, dFl, uFl, Ubr, dBr, Dfr, dfL)

The least common multiple of 7,7,3 and 15 is 105, and the cube indeed return to its original position after repeating FR 105 times. The above examples illustrates why repeating a specific operation finite times will always turn the cube back into its original position. In fact, the maximum number of repetition is 1260, with the operation RUUD'BD' (Thompson, 2024)[5]. This rule applies for not only 3 × 3 × 3 Rubik cubes but other Rubik cubes with different shapes or higher order. The next chapter will illustrate how pattern is generated on high order Rubik cubes.

5. Group Theory behind Pattern Generation

In this chapter, we use 7 × 7 × 7 Rubik cube as an example. The rules apply for any n × n × n Rubik cubes. For high order cubes, we slightly change the notation for each operation. Let Fi, Ui and Ri denote rotating the ith layer of the cube 90◦ clockwise. The figure below shows F1, F2 and F7 from left to right respectively.

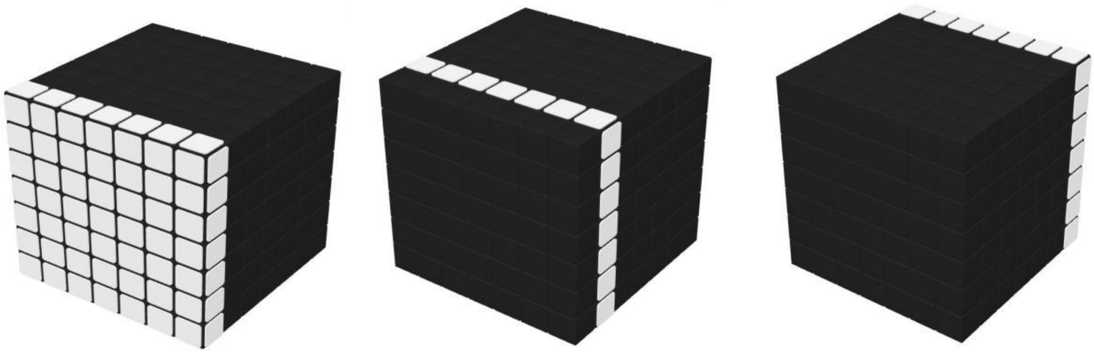

Figure 12. Layer rotation notation in high-order Rubik's Cubes

In pattern generation, we usually focus on the center blocks. To define each center block, establish a three-dimensional Cartesian coordinate system with the dfl corner of the cube as the origin. The block marked red below is defined as f(5, y, 3).

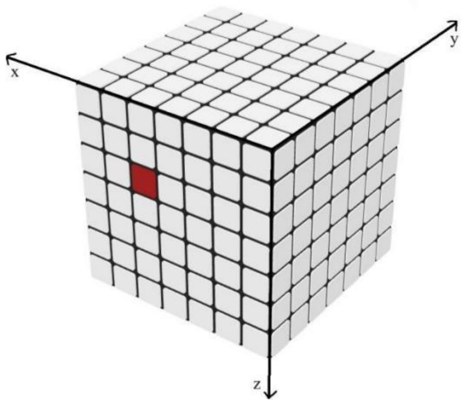

Figure 13. Cartesian coordinate system for defining center blocks

The first way of generating pattern is RiUjR'iU'j. The permutation for this operation is

(f(i, y, j), r(x, 8−i, j), d(i, 8−j, z))(u(i, j, z), b(i, y, j), l(x, 8−i, j))

The extension for this method is

RiRi+1 · · · Ri+kUjUj+1 · · · Uj+lR'iR'i+1 · · · R'i+kU'j U'j+1 · · · U'j+l

This transforms the rectangle with corner (i, i + k) and (j, j + l). The following operation generate a heart on the 7 × 7 × 7 cube, with white being the front face and red being the down face.

R3R5U2R'3R'5U'2
R2R3R4R5R6U3U4R'2R'3R'4R'5R'6U'3U'4
R3R4R5U5R'3R'4R'5U'5
R4U6R'4U'6

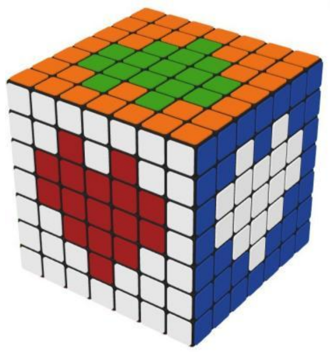

Figure 14. Heart pattern generated on a 7×7×7 Rubik's Cube using the first method

The second way is RiU1RjU'1R'iU1R'j. This operation swaps the block f(i, y, j) and u(i, 8−j, z). However, this operation affects edge blocks and corner blocks, so additional adjustments usu- ally are required. The extension for the first method also applies for this method, where two rectangles on the adjacent faces will be swapped location. Using this method, the following operation can also generate a heart, with red being the front face and white being the up face.

R3R5U1R2U'1R'3R'5U1R'2
U'1F'1
R2R3U'1R4R5U1R'2R'3U'1R'4R'5
U1F
R5R6U1R3R4U'1R'5R'6U1R'3R'4 U'1F1
R4U1R3U'1R'4U1R'3
U'1F'1
R3R4U1R5U'1R'3R'4U1R'5
U'1F1
R5U'1R3U1R'5U'1R'3 U1F'1 R4U1R6U'1R'4U1R'6
U'1
R4F'4R'4F4

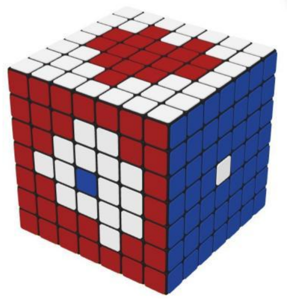

Figure 15. Heart pattern generated on a 7×7×7 Rubik's Cube using the second method

As shown, the second method is not as symmetric as the first method, and it cannot transform the block located at the very center. However, it is useful when only two blocks need to be swapped, where the first method swaps all corresponding blocks on the six faces.

6. Conclusion

This paper demonstrates how group theory provides a useful framework for understanding the structure and behavior of Rubik's Cube operations. By modeling cube moves as elements of a permutation group, it becomes possible to analyze the order of operations and explain why repeated algorithms eventually return the cube to its original configuration. The analysis of edge and corner permutations also reveals the constraints that determine the number of valid cube states. Extending these ideas to higher-order cubes, the study illustrates how specific sequences of layer rotations can be used to generate geometric patterns. These results show that mathematical concepts such as permutation groups play an important role in understanding both cube solving and pattern generation.

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